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ESTIMATION OF AN IMPLICIT ADDITIVE INDIRECT DEMAND SYSTEM[1]

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Preference structures in applied general equilibrium models are often limited to constant-elasticity-of-substitution (CES) forms due to the desire for global regularity. Hanoch (1975) uses indirect, implicit additive relationships—a generalization of the CES—to obtain more flexible demand relationships that are globally regular. These preference relationships unlink substitution effects from income effects in ways that go beyond relaxation of homotheticity, and are more flexible than their direct dual. However, the estimation of these models as demand systems has proven to be challenging, and most published work in this area has focused on estimation approaches that involve approximations or that cannot fully identify parameter values in the preference relationships. Our approach is direct, it avoids approximations, and it appears that parameters are identified. We demonstrate the estimation using the readily accessible Global Trade Analysis Project (GTAP) and the World Bank (International Comparison Program) databases, estimating the constant difference of elasticity or CDE directly in a maximum likelihood framework. In doing this, we show that the global regularity conditions stated in Hanoch (1975) can be slightly relaxed, and that the relaxed parametric conditions facilitate estimation. We introduce a normalization scheme that is beneficial for the scaling of the parameter values and which appears to have little impact on the *economic performance* of the estimated system.

Keywords: Consumer demand, Implicit utility, Parameter identification.

1. INTRODUCTION

Preference structures in applied general equilibrium models are often limited to constant-elasticity-of-substitution (CES) forms due to the desire for global regularity. Hanoch (1975) uses implicit additive relationships—a generalization of the CES—to obtain more flexible demand relationships that are globally regular. These models are parsimonious in the sense that the number of parameters to be identified is approximately proportional to the number of goods. Meanwhile, they are flexible enough so that goods are allowed to be substitutes/complements and normal/inferior (as introduced in Section 2). One of the key advantages of these implicit models is that their substitution matrices are much less restricted compared to standard CES models, where Allen-Uzawa partial elasticities of substitution can be expressed as functions of income elasticities, and there is

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always a ratio or additive relationship between them (Hanoch (1975); Houthakker (1960); Yang (2019)). The implicit additive models relax these restrictions on substitution effects, which can be expressed without reference to income effects, and thus effectively avoids the constraints on their relationships due to *Pigou's Law* (Barten (1977); Deaton (1974); Pigou (1910)). The implicit additive models can even reflect the case where there is no substitution among goods (Hanoch (1975)).

Recent papers have embedded these implicit preferences into general equilibrium models of economic growth and international trade, while estimating the preferences using additional constraints and rich datasets. For instance, Comin, Lashkari and Mestieri (2015) structurally estimate an implicit additive *direct* model using both household and macro-level data in the general equilibrium environment featuring structural change and economic growth; Yang (2019) constructs a general equilibrium gravity model of factor trade using the implicit additive *indirect* model and structurally identifies model parameters using geospatial and population data.

Most literature which directly addresses the problem of econometric identification of implicit additive demand models focuses on implicitly additive direct systems. For example, Rimmer and Powell (1996) construct An Implicitly Directly Additive Demand System (AIDADS) and estimate the model parameters using stochastic forms and maximum likelihood following an academic *mimeo* work by McLaren (1991). Their model can be viewed as a generalization of the Linear Expenditure System, and has been empirically studied by Cranfield *et al.* (2002). Preckel, Cranfield and Hertel (2010) develop a more generalized model based on the Rimmer and Powell (1996) implicit additive direct model applied in Gouel and Guimbard (2019) and Yang, Gouel and Hertel (2018).

Surprisingly, the advantage of implicitly additive indirect model over its direct dual has been ignored in most of the demand literature. Following Hanoch (1975) we stress that one important flexibility of the indirect forms, recently explored in Yang (2019), is the possibility of goods being complements even under global regularity restrictions. While such a feature is possible in the direct forms, they only hold locally and are incompatible with global regularity restrictions, because implicit additive direct models that are valid globally restrict all pairs of goods to be substitutes. This restriction on preferences for goods is unrealistic in general. Thus, it appears that the indirect, implicit additive models are more reasonable candidates for representing consumer preferences.

However, the direct estimation of implicit additive indirect models as demand systems has proven to be extremely difficult and most published work in this area has focused on approaches that involve approximations (Chen (2017); Hertel *et al.* (1991); Liu *et al.* (1998); Pudney (1981)). It appears that the reason for these approximations is due to the fact that utility is not observable. For this reason, these models have generally been estimated and used as production functions where the analogue to utilities is observable production output (see, e.g., Dar and Dasgupta (1985); Hashimoto and Heath (1995); Hawkins (1977);

Merrilees (1982); Surry (1993)). For the work that involves estimation of these models as demand systems, the demand literature suggests that any reduced-form approach requires double log-differencing to eliminate utilities, but would ultimately face parameter identification problems. Other empirical work on these models use entropy approaches and calculate the demand parameters from income and price elasticities of other available estimable demand systems such as the Linear Expenditure System proposed by Stone (1954) and the AIDADS family (Preckel, Cranfield and Hertel (2010); Rimmer and Powell (1996)), but these approaches are essentially calibration, rather than estimation.

We contribute to the literature by demonstrating the direct estimation of the indirect, implicit additive model as a demand system for the first time. Here we use the readily accessible GTAP (Aguiar, Narayanan and McDougall (2016); Aguiar *et al.* (2019)) and the World Bank (International Comparison Program or ICP) databases to develop a data set that takes expenditure and GDP data from GTAP and combines it with price data from the ICP. We then estimate an implicit additive indirect demand relationship, the constant difference of elasticity (CDE) directly in a maximum likelihood framework. In doing this, we provide some clarification of the global regularity conditions stated in Hanoch (1975) that result in conditions that facilitate estimation. In addition, we introduce a normalization scheme that identifies the parameters, is beneficial for the scaling of the parameter values, and that appears to have little impact on the economic performance of the estimated system.

2. THEORY OF THE IMPLICIT CDE FUNCTIONAL FORM

The demand model examined here is an implicit and indirect relationship that relates utility, prices and total expenditure as follows:

$$G(\frac{\mathbf{p}}{w}, u) = \sum_k \beta_k u^{e_k(1-\alpha_k)} (\frac{p_k}{w})^{1-\alpha_k} \equiv 1, \tag{2.1}$$

with $\log[u^{e_k}(p_k/w)]$ replacing $u^{e_k(1-\alpha_k)}(p_k/w)^{1-\alpha_k}$ in the limiting case where α_k approaches unity and where the subscript $k \in \{1, ..., N\}$ indexes commodities; with vector $\mathbf{p} = \{p_k\}_{k=1}^N$ denotes commodity prices; u denotes per capita utility, and w denotes the per capita total expenditure. The model parameters to be estimated, β's, e's, and α's are distribution, expansion and substitution parameters, respectively (Hanoch (1975)). In addition, the levels of per capita utility for each country are estimated. It is unusual to estimate utility in demand studies because it is unobservable; however, when an underlying explicit demand system can be derived from a utility maximization problem (e.g., in the CES case), the estimation produces everything needed to calculate utility up to a strictly increasing transformation. The difference lies in the fact that with an explicit functional form, there is no need to estimate utility, whereas with an implicit functional form utility must be explicitly estimated. As to the data, the prices are compiled from the ICP database for 2011; w is obtained from the GTAP database; and p_k/w may be interpreted as the unit-cost price or the normalized price of commodity k; and quantities, which do not appear in equation (2.1), but are nonetheless important. The stated parametric restrictions for the demand function to be globally valid (monotonic and quasi-concave) are that, at all $\mathbf{p}/w \gg 0$ (i.e., unit-cost prices all strictly positive), (i) $\beta_k, e_k > 0 \; \forall \; k \in N$, and (ii) either $\alpha_k > 1$ or $0 \leq \alpha_k \leq 1 \; \forall \; k \in N$. (The weak inequality in the second set of conditions in (ii) is justified in Section 4.3).

The model is categorized as an implicit (rather than an explicit) function because the equation defined by equation (2.1) cannot in general be algebraically solved for utility as an explicit function of exogenous variables and parameters. The model is indirect because its indifference curves, which illustrate demand patterns, are expressed in its unit-cost prices instead of quantities. The model is closely related to other standard demand models. For example, it is easy to show that, if we set $e_k = e = 1 \; \forall \; k \in N$ and $\alpha_k = \alpha \; \forall \; k \in N$, then equation (2.1) collapses to the standard indirect CES model.

By Roy's Identity, the derived demand correspondence is

$$q_k(\frac{\mathbf{p}}{w}, u) = \frac{w}{p_k} \frac{\theta_k}{\sum_j \theta_j} = \frac{w}{p_k} \Lambda_k, \tag{2.2}$$

where θ_k is an auxiliary variable such that

$$\theta_k = \beta_k u^{e_k(1-\alpha_k)} (1 - \alpha_k) (\frac{p_k}{w})^{1-\alpha_k}, \tag{2.3}$$

and Λ_k is the expenditure shares of goods k as a function of θ_k, which equals

$$\Lambda_k(\frac{\mathbf{p}}{w}, u) = \frac{\theta_k}{\sum_j \theta_j}. \tag{2.4}$$

The Allen-Uzawa elasticities of substitution σ_{km} are given by

$$\sigma_{km} = \alpha_k + \alpha_m - \sum_j \Lambda_j \alpha_j - \frac{\Delta_{km} \alpha_k}{\Lambda_k}, \tag{2.5}$$

where Δ_{km} is the Kronecker delta (equaling 1 if $k = m$; 0 if otherwise). Since σ_{km} is derived as a function of the share-weighted sum of expansion parameters, it can be negative (and thus complementary goods k and m may exist for $N \geq 3$) if the latter is large, or alternatively, if the expenditure share of goods k, which can possibly have a large substitution elasticity, is small.

The income elasticities η_k are given by

$$\eta_k = \frac{e_k(1-\alpha_k) + \sum_j \Lambda_j e_j \alpha_j}{\sum_j \Lambda_j e_j} + \alpha_k - \sum_j \Lambda_j \alpha_j. \tag{2.6}$$

It can be readily observed from equation (2.6) that goods are allowed to be inferior rather than normal, i.e., η_k can be negative. Again, this can happen if $\sum_k \Lambda_k \alpha_k$ is sufficiently large.

One other notable feature of this model is that the substitution matrix shown in (2.5) is much less restricted (and thus the implicit model is more flexible) compared to explicit models. Houthakker (1960) and Hanoch (1975) show that there are two types of tight linkages between substitution and income elasticities in the explicit cases. One is that the Allen-Uzawa substitution elasticities can always be derived as functions of the income elasticities, i.e., $\sigma_{km} = \eta_k \eta_m (\sum_j \alpha_j \Lambda_j)$ for the explicitly direct models, such as the CES model, or $\sigma_{km} = \eta_k + \eta_m (\sum_j \alpha_j \Lambda_j - 2)$ for the explicitly indirect models; and the other is that there is always a ratio or additive relationship between the substitution and income elasticities, i.e., $\eta_k/\eta_m = \sigma_{kj}/\sigma_{mj}$ for the direct case, or $\eta_k - \eta_m = \sigma_{kj} - \sigma_{mj}$ for the indirect case (Hanoch (1975); Yang (2019)).[1]

3. IDENTIFICATION ISSUES

There are two issues with identification related to the estimation of the CDE. The first applies to reduced-form approaches, where the transformations to eliminate utility from the demand system also eliminate the possibility of identifying all parameters in the system. The second is fundamental to the CDE functional form, and can be resolved by imposing normalizations as indicated in Section 4.1.

3.1. *Reduced-Form Approaches*

The previous demand literature suggests that any reduced-form approach (i.e., OLS regressions) for estimating these implicit indirect models as demand systems would require double log-differencing to eliminate utilities, but this would ultimately face identification problems. To see this, we take the natural log of

[1] In the literature of the 1960s, these relationships are sometimes referred to as relationships between cross-price derivatives and income derivatives, or substitution effects and Engel derivatives (see, e.g., Houthakker (1960); Powell (1966)).

both sides in equation (2.2) of the demand correspondence (with $\xi_k = p_k/w$):[2]

$$\begin{aligned}\ln q_k &= \ln\left[\beta_k u^{e_k(1-\alpha_k)}(1-\alpha_k)\xi_k^{-\alpha_k}\right] - \ln\left[\sum_j \theta_j\right] \\ &= \ln[\beta_k(1-\alpha_k)] + e_k(1-\alpha_k)\ln u - \alpha_k \ln \xi_k - \ln\left[\sum_j \theta_j\right].\end{aligned} \tag{3.1}$$

Eliminating the last term in (3.1) by using the logarithmic ratio:

$$\begin{aligned}\ln\frac{q_k}{q_1} &= \ln\frac{\beta_k(1-\alpha_k)}{\beta_1(1-\alpha_1)} + [e_k(1-\alpha_k) - e_1(1-\alpha_1)]\ln u - \alpha_k \ln\xi_k + \alpha_1 \ln \xi_1 \\ &= A_k + Z_k \ln u - \alpha_k \ln \xi_k + \alpha_1 \ln \xi_1 + \epsilon_k,\end{aligned} \tag{3.2}$$

where $A_k = \ln\{\beta_k(1-\alpha_k)/[\beta_1(1-\alpha_1)]\}$ and $Z_k = e_k(1-\alpha_k) - e_1(1-\alpha_1)$; q_1 is the chosen good for normalization; ϵ_k $\forall$ $k \in N-1$ is assumed to be the random error, which is independent of ξ_k, and has mean zero and constant variance.

Note that the estimation equation (3.2) is suitable for the function used as a production function, such as in Surry (1993) where u is the observable level of output (rather than utility). Because the cardinal value of utility is unknown, we cannot directly estimate (3.2) by performing a regression of the logarithmic ratio of quantities on the explanatory variables on the right-hand side. In order to estimate (3.2) as a demand function, we need to first eliminate the unobservable u. One way to do this is to choose q_2 as a second good for normalization:

$$\begin{aligned}\ln\frac{q_2}{q_1} &= \ln\frac{\beta_2(1-\alpha_2)}{\beta_1(1-\alpha_1)} + [e_2(1-\alpha_2) - e_1(1-\alpha_1)]\ln u - \alpha_2 \ln\xi_2 + \alpha_1 \ln \xi_1 \\ &= A_2 + Z_2 \ln u - \alpha_2 \ln \xi_2 + \alpha_1 \ln \xi_1 + \epsilon_2,\end{aligned} \tag{3.3}$$

where $A_2 = \ln\{\beta_2(1-\alpha_2)/[\beta_1(1-\alpha_1)]\}$ and $Z_2 = e_2(1-\alpha_2) - e_1(1-\alpha_1)$.

Isolating (3.3) so that only u is on the left-hand side:

$$\ln u = \frac{-A_2 + \alpha_2 \ln \xi_2 - \alpha_1 \ln \xi_1 + \ln\frac{q_2}{q_1} - \epsilon_2}{Z_2}. \tag{3.4}$$

Now substituting (3.4) into (3.2) to eliminate u without loss of generality in

[2] Hanoch (1975) illustrates steps of generating the estimation equation as a production function using the log-differencing; it is important to show that the reduced-form approach will not solve the estimation equation as a demand function where u is not observable, which gives the merit of our direct maximum likelihood approach introduced in Section 4.

terms of the functional form, with $R_k = Z_k/Z_2$:

$$
\begin{aligned}
\ln\frac{q_k}{q_1} &= A_k + \frac{Z_k}{Z_2}(-A_2 + \alpha_2 \ln \xi_2 - \alpha_1 \ln \xi_1 + \ln\frac{q_2}{q_1} - \epsilon_2) \\
&\quad - \alpha_k \ln \xi_k + \alpha_1 \ln \xi_1 + \epsilon_k \\
&= A_k - R_k A_2 + R_k \alpha_2 \ln \xi_2 - R_k \alpha_1 \ln \xi_1 + R_k \ln \frac{q_2}{q_1} \\
&\quad - \alpha_k \ln \xi_k + \alpha_1 \ln \xi_1 - R_k \epsilon_2 + \epsilon_i.
\end{aligned}
\tag{3.5}
$$

By rearranging equation (3.5), we now obtain an estimation equation as follows:[3]

$$
\ln\frac{q_k}{q_1} = S_k - \alpha_k \ln \xi_k + R_k \alpha_2 \ln \xi_2 + (1 - R_k)\alpha_1 \ln \xi_1 + R_k \ln \frac{q_2}{q_1} + \varphi_k, \tag{3.6}
$$

where $\varphi_k = \epsilon_k - R_k \epsilon_2$ is the error term $\forall\ k \in N-2$, satisfying standard assumptions as for ϵ_k.

Note that the regression estimation to equation (3.6) immediately yields the intercept S_k, and coefficients α_k, $R_k\alpha_2$, $(1-R_k)\alpha_1$ and R_k, which automatically yields α_1 and α_2. Given R_k, the relationship between e_k and e_2 can be identified. Therefore, the system can be completely solved if and only if β_2 and e_2 are pinned down, given S_k, which cannot be accomplished without further identities. With this approach, it is clear to see that only $N-2$ substitution parameters α_k can be estimated, but we cannot solve for expansion parameters e_k and distribution parameters β_k. Thus, given the model specification, the reduced form estimation framework is not well-suited for identification. Our method (introduced in Section 4), while requiring more computational effort, is a practical approach relying on structural optimization that allows us to identify all key parameters of the model.

3.2. *Excess Degrees of Freedom*

Since u cannot be observed, the preference defining relationship in equation (2.1) has multiple sets of parameter values that will satisfy the relationship equally well. To see this, note that since any strictly increasing transformation of utility will not alter the ordering of preferences for alternative consumption bundles, such transformations will have no impact on the quality of the estimated

[3]In practice, empirical econometric work often concerns the simultaneity bias issues (i.e., income which equates total expenditure is jointly determined by the quantity demanded for goods k) and the fact that the arbitrary choice of normalized goods can lead to multiple parameter estimates at length N (see also Surry (1993)). In this case, the estimation equation can be modified to: $\ln \Lambda_k/\Lambda_1 = S_k + (1-\alpha_k)\ln \xi_k + R_k\alpha_2 \ln \xi_2 + [(1-R_k)\alpha_1 - 1]\ln \xi_1 + R_k \ln q_2/q_1 + \varphi_k$ (by letting $\Lambda_k = q_k(p_k/w) = q_k\xi_k$ so that $\ln q_k/q_1 = \ln \xi_k + \ln \xi_1$). This, however, raises isomorphic identification issues as from equation (3.6).

relationship. For example, consider the transformation $u = \rho\upsilon^{\delta}$ where $\upsilon > 0$, which will be strictly increasing if $\rho > 0$ and $\delta > 0$. In this case, υ will be as good as u for explaining the data. Substituting the transformed u into equation (2.1):

$$
\begin{aligned}
1 \equiv G\Big(\frac{\mathbf{p}}{w}, u\Big) &= \sum_k \beta_k u^{e_k(1-\alpha_k)} \Big(\frac{p_k}{w}\Big)^{1-\alpha_k} \\
&= \sum_k \beta_k \big[\rho\upsilon^{\delta}\big]^{e_k(1-\alpha_k)} \Big(\frac{p_k}{w}\Big)^{1-\alpha_k} \\
&= \sum_k \beta_k \rho^{e_k(1-\alpha_k)} \upsilon^{\delta e_k(1-\alpha_k)} \Big(\frac{p_k}{w}\Big)^{1-\alpha_k} \\
&= \sum_k \widetilde{\beta}_k \upsilon^{\widetilde{e}_k(1-\alpha_k)} \Big(\frac{p_k}{w}\Big)^{1-\alpha_k},
\end{aligned} \tag{3.7}
$$

where $\widetilde{\beta}_k = \beta_k \rho^{e_k(1-\alpha_k)}$, and $\widetilde{e}_k = \delta e_k$. Since ρ and υ were any strictly positive constants, there is a continuum of values for the $\widetilde{\beta}_k$ and $\widetilde{e}_k$ that exactly satisfy equation (2.1) given that this defining relationship is satisfied for β_k and e_k. Thus, these parameters are not fully identified. For this reason, we introduce normalizations for these two sets of parameters, choosing our normalizations in such a way that the estimated parameter values are "well-scaled" as we will illustrate in Section 4.1.

4. ECONOMETRIC PROCEDURE

We follow the maximum likelihood estimation procedure for implicit additive direct demand systems used by others (Cranfield *et al.* (2002); Gouel and Guimbard (2019); Preckel, Cranfield and Hertel (2010); Yang, Gouel and Hertel (2018)). In this framework, we estimate the implicit indirect relationship using constrained maximum likelihood subject to a system of equations and parametric constraints. The log-likelihood function is given by

$$
\log L = -0.5I[J(1 + \log 2\pi) + \log |\mathbf{R'R}|], \tag{4.1}
$$

where I and J are the numbers of countries (or regions) and goods, respectively; $|\mathbf{R'R}|$ is the determinate of cross-goods error covariance matrix. As in Gouel and Guimbard (2019) based on similar concerns of heterogeneity in the cross-country consumption patterns and potential heteroscedasticity, our added measurement errors (expressed in d's) are in quantities, instead of budget shares:

$$
d_{ik} = q_{ik} - \widehat{q}_{ik} = \frac{w}{p_{ik}}(\Lambda_{ik} - \widehat{\Lambda}_{ik}), \tag{4.2}
$$

where i and k index countries and goods, respectively; $\widehat{q}_{ik}$ and $\widehat{\Lambda}_{ik}$ are the fitted quantities and expenditure shares, and the components of $\mathbf{R}$, r_{nm}, are constrained by:

$$\sum_n r_{nk} r_{nm} = \sum_i \frac{d_{ik} d_{im}}{I}, \tag{4.3}$$

along with $r_{km} = 0 \; \forall \; m > k$ making $\mathbf{R} = [r_{km}]_{k,m=1,\dots,N}$ an upper triangular Cholesky factorization of the error covariance matrix. The advantage of working with this factorization of the cross-equation error covariance matrix is that evaluation of the determinate of the covariance matrix is simple—it is the square of the product of the diagonal elements of $\mathbf{R}$.

4.1. *Identification and Normalization Strategy*

As shown earlier the system and the model parameters are not fully identified without removing excess degrees of freedom from the parameter space. It is useful to first consider a less general implicit indirect non-homothetic CES demand system:

$$G\Big(\frac{\mathbf{p}}{w}, u\Big) = \sum_k \beta_k u^{e_k(1-\alpha)} \Big(\frac{p_k}{w}\Big)^{1-\alpha} \equiv 1, \tag{4.4}$$

which can be obtained by restricting $\alpha_k = \alpha \; \forall \; k$ in equation (2.1). This system was studied by Comin, Lashkari and Mestieri (2015),[4] who developed an approach to estimation that influences ours, as well as normalizations of parameter space that are similar in spirit to ours.

Comin, Lashkari and Mestieri (2015) observe that the expressions for the own price and income elasticities of demand are invariant to a multiplicative scaling of the parameters equivalent to our β_k and e_k for the direct non-homothetic CES case. In the interest of parameter identification, they remove a degree of freedom for each of these sets of parameters by normalizing these parameters to unity for one good (e.g., $k = 1$). We show in Section 3.2 the same invariance to rescaling of these parameters in the indirect case. We choose a slightly different normalization scheme by setting $\sum_k \beta_k \equiv 1$ and $\sum_k e_k \equiv 1$, again removing one degree of freedom for each of these sets of parameters. Our normalization of the β_k combined with the form of our implicit preference defining relationship (2.1) will tend to improve the scaling of the terms $u^{e_k(1-\alpha_k)}(p_k/w)^{1-\alpha_k}$, which we have observed tends to improve the scaling of the parameters α_k. Similarly, the choice

[4] The system specified by Comin, Lashkari and Mestieri (2015) is $\sum_k (\Omega_k C^{\epsilon_k})^{\frac{1}{\sigma}} C_k^{\frac{\sigma-1}{\sigma}} \equiv 1$, where C_k denotes demand, C is an aggregator index, Ω_k, σ, and ϵ_k are parameters. This can be viewed as the direct form of our relationship in (4.4).

to normalize the sum of the exponents e_k to be equal to one on average across the goods tends to improve the scaling of the u levels. Our motivation, as well as that of Comin, Lashkari and Mestieri (2015), is to obtain parameter identification for our demand system. While our normalizations are mathematically equivalent, our strategy of normalizing sums of these parameters rather than individual ones avoids the possibility of making an unfortunate choice for the parameters that are set to one, which may serve to make the other parameters either large or small numbers. Given that we solve our estimation problems numerically using general nonlinear programming software, attention to scaling can improve our likelihood of success in estimation.

4.2. *Constrained Optimization*

The constrained optimization problem is to maximize equation (4.1), subject to the constraints (4.2), (4.3) along with $r_{km} = 0\ \forall\ m > k$, the normalized implicit indirect additivity relationship (4.4), the normalization equations $\sum_k \beta_k \equiv 1$ and $\sum_k e_k \equiv N$, the auxiliary identities (2.2)–(2.4), as well as the *redefined* parametric restrictions of the demand system.

We find that, while strict inequalities imposed on α's are sufficient and necessary conditions for implicitly direct relationships where the isoquant reflects the choices of quantity of goods (e.g., Constant Ratio Elasticity of Substitution or CRES Model) and corner solutions may be empirically observed from the data, it is not a necessary condition for the implicit indirect function where the isoquant reflects the normalized price of goods. Therefore, we impose weak inequalities for the lower bounds on the α's and our parametric restrictions are (i) $\beta_k, e_k > 0$ $\forall\ k \in N$, and (ii) $0 \leq \alpha_k \leq 1\ \forall\ k \in N$ (see Section 4.3). This choice ignores the case where $\alpha_k > 1$, which we found not to be relevant for our dataset because the resulting elasticity estimates were not credible. As in Hanoch (1975), we interpret $(p_k/w)^{1-\alpha_k}$ when $\alpha_k = 1$ as $\ln(p_k/w)$, although in our empirical work it turned out that $\alpha_k < 1$ for all k.

4.3. *Proof of Global Regularity*

In reviewing the published regularity conditions in Hanoch (1975) we noticed what appear to be some minor discrepancies in the conditions related to the boundaries for the parameter α_k. While the ranges for α_k that are required for global regularity are consistent, the inclusion of the end points of the ranges is not as clear.[5] Thus in this section, we readdress the proof of global regularity

[5]In Hanoch (1975) (p. 403), with different subscript i, the stated parametric restrictions for $d_i(= 1 - a_i)$ in the CRES Model (i.e., equation (2.16)) is that either $0 < d_i < 1$ or $d_i \leq 0$, $\forall$ i, and for a_i is that either $a_i > 1$ or $0 < a_i \leq 1\ \forall\ i$; then on p. 411, the stated conditions for $b_i(= 1 - \alpha_i)$ in the CDE Model (i.e., equation (3.15)) is that either $0 < b_i < 1$ or $b_i \leq 0\ \forall\ i$, and for α_i is that either $\alpha_i \geq 1$ or $0 < \alpha_i < 1\ \forall\ i$. Because d_i's and b_i's as well as a_i's and α_i's are used interchangeably, the same regularity conditions for d_i and b_i should imply the same restrictions for a_i and α_i which, however, are not what we find in Hanoch (1975).

with an eye to whether the extreme value $\alpha_k = 0$ for some k should be included.

A preference relationship is said to be globally regular if the relationship in equation (2.1) is monotone and satisfies a quasi-concavity property. To show this, it is useful to begin from the direct form of the CDE, the CRES. We demonstrate that under a mildly relaxed set of parametric restrictions relative to Hanoch (1971, 1975), that we obtain global regularity for the CRES. Hanoch (1975) argues that regularity of the direct CRES relationship yields regularity of the indirect CDE relationship due to symmetry between $f(\mathbf{x})$ and its *indirect reciprocal* $g(\mathbf{p}/w)$ in the CDE.

Following Hanoch (1975), we begin by setting up the cost minimization problem for the CRES subject to the defining constraint for the CRES relationship. This relationship has near-identical form to the CDE as follows:

$$\mathcal{H}(\mathbf{x}, u) = \sum_k \beta_k u^{-e_k(1-\alpha_k)}(x_k)^{1-\alpha_k} \equiv 1, \tag{4.5}$$

with $\log(x_i/u^{-e_k})$ replacing $u^{-e_k(1-\alpha_k)}(x_k)^{1-\alpha_k}$ in the limiting case where α_k approaches unity and where $\mathbf{x}$ denotes the levels of inputs. The stated regularity conditions are: (i) $\beta_k, e_k > 0 \ \forall \ k \in N$, and (ii) either $\alpha_k > 1$ or $0 < \alpha_k \leq 1 \ \forall \ k \in N$.

THEOREM 4.1 *The latter set of regularity conditions in equation (4.5), i.e., condition (ii): either $\alpha_k > 1$ or $0 < \alpha_k \leq 1 \ \forall \ k \in N$, can be relaxed to either $\alpha_k > 1 \ \forall \ k \in N$, or $0 \leq \alpha_k \leq 1 \ \forall \ k \in N$, such that the formulated constrained optimization program (4.6) is still a convex program.*

PROOF: The proof of monotonicity is straightforward (see Appendix A). To see that $\mathcal{H}(\mathbf{x}, u)$ is quasi-concave, using (4.5) as our preference defining relationship, the cost minimization problem for fixed u may be formulated as:

$$\begin{aligned} \underset{x_k \forall k \in N}{\text{minimize}} \quad & \sum_k c_k x_k \\ \text{subject to:} \quad & \sum_k \beta_k u^{-e_k(1-\alpha_k)}(x_k)^{1-\alpha_k} - 1 \equiv 0 \\ & x_k \geq 0. \end{aligned} \tag{4.6}$$

where c_k is the cost of per unit of good k.

In contrast to Hanoch (1971), we are explicit about the non-negativity of x_k because the parametric relaxations we introduce give rise to the existence of corner solutions when $\alpha_k = 0$ for some k. Hanoch (1971) notes that the terms $\{\beta_k(1 - \alpha_k)\}$ must all be of the same sign, a restriction we maintain. In the case where these exponents are non-negative and at least one is positive, the Lagrange multiplier on the equality constraint must be negative, and the equality in the constraint can be replaced by less than or equal to relationship. Without loss of generality, we can choose to number the arguments such that, for the first set of $x_k \ \forall \ k \in \{1, ..., S\} : 0 < \alpha_k < 1$, and for the second set of $x_k \ \forall \ k \in \{S+1, ..., K\} : \alpha_k = 0$.

Hanoch (1971) argues that the degree S principal minor of this Hessian is a negative semi-definite (NSD) matrix. The selected values for α_k are such that all of the second order derivatives in the rows and columns of the Hessian are equal to zero. In the case where $S < k \leq K$, x_k appears linearly in the defining constraint. Thus, the full Hessian matrix is NSD and the constraint function is concave and thus defines a convex feasible region. Since the objective and all other constraints in the problem are linear, (4.6) is a convex program, and the set of optimal solutions for this problem is convex. This demonstrates that the relationship defined by the constraint in (4.6) satisfies the quasi-concavity property needed for global regularity of the preference relationship. The argument for the case where $\alpha_k \geq 1 \ \forall \ k \in N$ proceeds as in Hanoch (1975). *Q.E.D.*

This leads us to our relaxed set of parametric conditions for the CRES: $\beta_k, e_k > 0 \ \forall \ k \in N$, and either $\alpha_k > 1 \ \forall \ k \in N$, or $0 \leq \alpha_k \leq 1 \ \forall \ k \in N$. Again, following Hanoch (1975), due to *complete formal symmetry* between the direct and indirect cases, global regularity of the CRES in $\mathbf{x} \gg 0$ implies global regularity of the CDE in $\mathbf{p}/w \gg 0$. Note that the corner solutions for the CRES do not similarly imply that the CDE will generate corner solutions when some of our parametric restrictions on α_k are binding. This is because the envelope conditions used to reclaim the demand quantities involve not only derivatives with respect to the numerator in p_k/w, but also in the denominator, which appears in other terms in our defining equation.

5. DATA

The data used in the estimation come from the GTAP version 9.2 (Aguiar, Narayanan and McDougall (2016)) and the World Bank's ICP database with the reference year 2011.[6] The original database in GTAP covers 141 countries (or geographic regions) and 57 sectors. For the estimation, we reduce the sample size to 121 GTAP countries for the purpose of mapping with what are available within the set of 202 ICP countries. For the income and price elasticities, we access the latest GTAP 10 database (Aguiar *et al.* (2019)) with the same reference year and use the estimated parameter values to calculate the income and price

[6]The reason for choosing the GTAP database is twofold—one, the GTAP compiles and maintains a reliable reconciled database originated from prominent data sources such as the World Bank, UN Statistics Division as well as the CIA World Factbook, etc.; two, the GTAP Model has more than 20,000 users around the world and we are in the hope that our direct estimates for the first time will benefit a large community. We combine the two data sources because the ICP data reflects greater price variation, while allowing us to map with the demand categories at aggregate levels. Such a feature is important for our estimation (as discussed later in this section).

(compensated or uncompensated) elasticities for 141 countries. This is because the requirements for computing these elasticities only include prior knowledge of *fixed* parameters and expenditure shares.

Demand theory suggests that the CDE demand model and, in general, any other implicitly additive models are more appropriate for goods defined at aggregate levels. These models imply that their Allen-Uzawa partial elasticities of substitution between goods k and m are always proportional to their derived substitution functions governing single-good k's substitution characteristics (Barten (1977); Hanoch (1975)). For this reason, we map the 57 GTAP sectors to 10 aggregate commodities. Figure 1 on page 14 shows the empirical patterns of evolution of expenditure shares across 141 countries and 10 broad commodities. We rank these countries with per capita GDP (calculated using per capita real expenditure) from low to high and observe that expenditures on grain-based foods and livestock decline as income increases, whereas the shares of spending on *HousOthServ* (housing, education, health, and public services) and *FinService* (financial services) follow the opposite trend.

The total nominal expenditure is aggregated by the value of domestic and import purchases by private households. The per capita consumption and expenditure shares are obtained by dividing the total nominal expenditure by population and total expenditures, respectively. The estimation of the CDE relies heavily on a good domestic price proxy that is able to reflect realistic price variations. For this reason, we choose to adopt the ICP prices instead of using tariffs as a proxy (which are calculated by dividing the value of tradable commodities at importer's market prices by their world cost, insurance and freight or CIF prices.) to define price variation. These are calculated using 2011 Purchasing Power Parity (PPP) adjusted by exchange rates across regions for the same year.[7]

6. ESTIMATION RESULTS

Table I on page 15 reports the parameter estimates of the CDE demand system. These are the parameter values that maximize the likelihood function (4.1). We formulate the estimation as a mathematical programming problem using the General Algebraic Modeling System (GAMS) version 28.2.0 with the CONOPT nonlinear programming (NLP) solver (Drud (1985)) on a Windows 64-bit operating system. Our estimates are robust as we demonstrate in the discussion of our identification strategy later in this section.

6.1. *Likelihood Testing and Parameter Scales*

Previous empirical framework of implicitly direct demand systems, such as Cranfield *et al.* (2002) and Preckel, Cranfield and Hertel (2010), note that the

[7]One important difference between ICP and GTAP data on prices is the price margin categories. The ICP contains wholesale, retail and transportation margins altogether, which are treated separately in GTAP. See Reimer and Hertel (2003) for details.

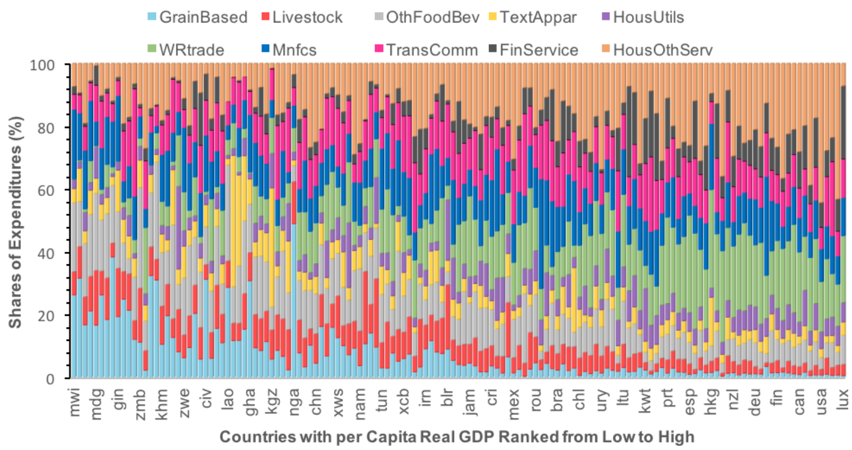

FIGURE 1.— Expenditure Shares across 141 Countries and 10 Aggregate Commodities (from the data as described in Section 5).

right-hand side of the implicit additivity defining equation can be any constant $\mathcal{M} \in \mathbb{R}$ and that it can be estimated. They proceed to estimate its numerical value. Given the excess degrees of freedom demonstrated in equation (3.7), it can be readily seen that equation $\sum_k \beta_k = 1$ is a *de facto* normalization where $\mathcal{M} \equiv 1$. In the interest of good scaling of the individual terms in the sum in the defining equation (4.4), we run a series of likelihood value testing by choosing to set:[8]

$$G(\frac{p_k}{w}, u_i) \equiv \sum_k \beta_k \equiv \mathcal{M} \in \mathbb{R}_{++} \quad \forall\, \beta_k > 0. \tag{6.1}$$

6.1.1. *Value of the Log-Likelihood Function*

The results reported in Table I are for $\mathcal{M} \equiv 1$, where its maximized objective value of the log likelihood function is 4853.194. We find that, as we increase the value of the joint right-hand sides of the sum of β's and the defining constraint, the value of the objective (log-likelihood) function unambiguously increases. That is, as $\mathcal{M}$ is increased from 1 to 10 to 100 to 1,000 to ∞, the objective value also increases, but the change (growth rate) in the likelihood becomes progressively smaller. For instance, for the factor of 10 increases from 1 to 10, the increase in the likelihood value is about 0.21%; for the increase from 10 to 100, the likelihood

[8]The parametric restrictions in this model governs that $\mathcal{M}$ must be strictly positive, since β_k's are strictly positive.

TABLE I

Estimated substitution, distribution and expansion parameters of the implicitly indirect function across 10 aggregate commodities.

Commodities	α's	β's	e's
GrainBased	0	4.950E-5	0.748
Livestock	0.567	0.002	0.588
OthFoodBev	0.858	0.056	0.710
TextAppar	0.931	0.321	1.189
HousUtils	0.886	0.055	0.482
WRtrade	0.754	0.104	1.383
Mnfcs	0.849	0.178	1.222
TransComm	0.871	0.118	0.745
FinService	0.819	0.144	1.565
HousOthServ	0.518	0.022	1.368

increase is about 0.13% and for the increase from 100 to 1,000, the likelihood is less than 0.1%.

In parallel, the scaling of the model parameters suffers with the magnitude of $\mathcal{M} \in \mathbb{R}_{++}$. As $\mathcal{M}$ approaches infinity with each time increasing by a factor of 10 to ultimately achieve 100,000, algorithms for solving constrained NLP unsurprisingly require an increasingly large number of iterative steps in the parameter space (provided by the nonlinear implicit relationships) to seek an optimal solution, while the scaling of the problem becomes incrementally worse and β's eventually become extreme across goods.[9] However, if one examines the economic properties of the demand system, the elasticities of substitution and income elasticities do not change appreciably.

6.1.2. *Testing for Income Elasticity of Demand*

In this respect, we calculate average income elasticities across commodities and find that, as $\mathcal{M}$ increases from 1 to 100,000, the absolute percentage changes in income elasticities from the base where $\mathcal{M} \equiv 1$ for most goods are less than 5% (with changes in food commodities being the highest which is around 20% when approaches 100,000). The changes in average income elasticities, however, fall within the range of 3.7% to 6.7% relative to the elasticities based on the $\mathcal{M} \equiv 1$ estimated values (see Table II). Still, it should also be noted that the average income elasticity estimated with $\mathcal{M} \equiv 1$ is only about 0.985 and those percentage changes from base estimates have small impacts on the income elasticities.

[9] To reduce the computational burden in the initial phase of minimizing infeasibility, we first estimate a parameterized implicit non-homothetic CES model by setting $\alpha_k = \alpha \; \forall \; k$ (i.e., an implicit NHCES Model; see also Comin, Lashkari and Mestieri (2015)). Yang (2019) shows this parameterization and that the *quasi* Marshallian correspondence of the parameterized indirect case is identical to the direct case in Comin, Lashkari and Mestieri (2015). This procedure is effective for finding starting values that are feasible to characterize the implicit indirect relationships in the estimation of the CDE.

6.1.3. *Testing for Own-Price Elasticity of Demand*

We examine the changes in price elasticities, finding that moving $\mathcal{M} \in \mathbb{R}_{++}$ away from unity has little impact on economic behavior. Initial own-price elasticity values are small with small percentage changes (even as $\mathcal{M}$ approaches 100,000). For example, changes in compensated (uncompensated) own-price elasticities for non-food commodities are less than 15% (10%) as $\mathcal{M}$ increases from 1 to 100,000. The percentage changes are relatively larger for food commodities. However, their initial base values are quite small and thus the absolute changes from the base value of the joint right-hand sides where $\mathcal{M} \equiv 1$ are minor in terms of their impacts on economic performance. GrainBased for example has an average compensated (uncompensated) own-price elasticity of about 0.05 (0.108). As $\mathcal{M}$ increases to 100,000, the absolute change in compensated (uncompensated) own-price elasticities of GrainBased is only about 0.008 (0.018).

TABLE II

Percent Changes in Income Elasticities as the Value of the Joint Right-Hand Sides of the Sum of β's and the Defining Constraint Increases for the Factor of 10 from Unity to 100,000.[1]

Commodities	$\mathcal{M} \equiv 10$	$\mathcal{M} \equiv 100$	$\mathcal{M} \equiv 1,000$	$\mathcal{M} \equiv 10,000$	$\mathcal{M} \equiv 100,000$
	Log-Likelihood Value				
	4863.4	4870.2	4874.0	4876.3	4877.8
GrainBased	7.2	12.7	16.8	19.1	19.4
Livestock	12.8	17.7	19.8	20.3	20.3
OthFoodBev	0.6	0.2	1.0	1.5	1.6
TextAppar	-3.0	-3.4	-3.4	-3.3	-3.4
HousUtils	0.0	1.4	2.5	3.2	3.3
WRtrade	-1.1	-3.3	-5.0	-4.9	-3.4
Mnfcs	-4.0	-4.8	-4.9	-4.8	-5.0
TransComm	-0.6	0.0	0.7	1.1	1.0
FinService	3.5	0.8	-3.2	-4.4	-3.8
HousOthServ	-1.2	-2.4	-3.0	-4.2	-5.3
Average[2]	3.4	4.7	6.0	6.7	6.7

[1] Percentage changes are calculated relative to initial income elasticities where $\mathcal{M} \equiv 1$.
[2] Average percentage changes calculated based on the absolute percentage changes across goods.

Finally, as $\mathcal{M}$ continues to increase, the computational effort required to estimate the system increases substantially and leads to occasional software failures. Thus, in the interest of defining an estimation procedure that is computationally efficient, we choose $\mathcal{M} \equiv 1$, again noting that this does not appear to have important consequences for the economic properties of our preference relationship.

6.2. *Robustness of Parameter Estimates*

In considering the potential for identification of the parameters of our preference relationship, we were able to demonstrate that at least two normalizations were required to remove excess degrees of freedom from the system. A reasonable follow-on question might be, are there additional normalizations that could be added? To this end, we would like to know whether the same log likelihood level can be achieved with different configurations of parameter values. While this is an exceedingly difficult question to answer absent an assurance that the problem is a convex program, we ask this question starting from the estimated parameter solution.

Operationally, we proceed as follows. First, we estimate the parameters based on the problem defined by equations (4.1)-(4.4), auxiliary identities (2.2)-(2.4), parametric restrictions in Section 4.1, and the normalizations $\sum_k \beta_k = 1$ and $\sum_k e_k = N$. We then construct a new problem that includes the relationships in the original estimation problem plus the requirement that the value of the likelihood function must be at least as great as the maximum obtained when estimating the parameters. With that system of constraints, we formulate a series of new problems with objectives that in turn maximize and minimize each individual parameter. For example, we define a new objective variable z such that $z \equiv \alpha_{Mnfcs}$ and solve two problems—one that maximizes z and one that minimizes z. All other parameters are allowed to change to accommodate a change in z subject to the constraints. If the maximum absolute change in z relative to the original estimate of the parameter is essentially zero, then we conclude that it is locally impossible to change that parameter without decreasing the likelihood function. This procedure was repeated for all model parameters, α_k, β_k, e_k, and the estimated level of utility for each country. Table III on page 17 reports a sample of our testing results as we maximize or minimize $z \equiv \alpha_{Mnfcs}$.

TABLE III
ABSOLUTE CHANGES IN THE PARAMETER VALUES AS $z \equiv \alpha_{Mnfcs}$ IS MAXIMIZED OR MINIMIZED.

Commodities	α's		β's		e's	
	max	min	max	min	max	min
GrainBased	0.000	0.000	0.000	0.000	0.000	0.001
Livestock	0.000	0.000	0.000	0.000	0.000	0.000
OthFoodBev	0.000	0.000	0.000	0.001	0.000	-0.001
TextAppar	0.000	0.000	-0.001	0.000	0.001	0.001
HousUtils	0.000	0.000	0.000	0.000	0.000	0.000
WRtrade	0.000	0.000	0.000	0.000	0.000	0.000
Mnfcs	0.000	-0.001	0.002	0.000	0.000	0.001
TransComm	0.000	0.000	-0.001	0.000	0.000	0.000
FinService	0.000	0.000	0.000	0.000	0.000	0.000
HousOthServ	0.000	0.000	0.000	0.000	-0.001	-0.001

We find that the room for moving most of the parameters is on the order of 0.001 (0.01 at most), which is in the range of a tenth of a percent to one percent. These results indicate that the parameter values cannot be changed substantially without reducing the likelihood value or violating the problem constraints, suggesting that no additional parameter normalizations are needed.

6.3. *Income Elasticities*

Figure 2 illustrates the evolution of the calculated income elasticities across 10 broad commodities with logarithm of GDP (PPP) per capita. For illustration purposes, we select 4 representative countries (Rwanda, China, Brazil and the USA) and indicate their real income levels in the figure. Our empirical evidence based on the estimated CDE parameters suggests a remarkable non-linear relationship between changes in income and goods consumption across income levels specified in the figure. Here, it should be noted that with the implicit relationship manifested by the CDE functional form the income elasticities are functions of expenditure shares, and the expenditure shares, in turn, are functions of implicit utilities that cannot be substituted out cleanly as in the CES model where the indifference curves and the income expansion path have exactly the same slope (i.e., utility is linear in income).[10]

All of our calculated income elasticities (while negative values are theoretically possible) are positive, indicating that all goods are normal goods. As expected, the income elasticities of demand for foods (GrainBased or Livestock) are all less than one across the income spectrum. This is because, as income (and total expenditure) rises, the expenditure shares for food decline as shown in Figure 1. It also shows that the income elasticities are significantly higher for foods that are more *aspirational* and less *necessary*. This is not only true when we compare basic foods such as GrainBased and Livestock with those which are less basic such as OthFoodBev (i.e., processed and higher-end foods, beverage and tobacco), but also valid for comparisons between GrainBased and Livestock *per se*. A few goods have income elasticities greater than one and that are considerably higher than the income elasticities of demand for others. Those are the goods which are deemed to be luxuries such as HousOthServ, FinService and WRtrade (i.e., wholesale and retail trade, motor vehicles services and repairs).

7. CONCLUSION

Preference structures in applied general equilibrium models are often limited to constant-elasticity-of-substitution or CES forms due to the desire for global regularity. Hanoch (1975) uses implicit, additive relationships, that can be viewed as a generalization of the CES, to obtain more flexible demand relationships. These preference relationships unlink substitution effects from income effects in ways that go beyond relaxation of homotheticity. However, the estimation of these models as demand systems has proven to be difficult and most published work in this area has focused on approaches that involve approximations. Here we use the GTAP and the World Bank ICP databases to estimate an implicit indirect

[10] In the standard (explicit) CES model, if we define a price index of aggregate demand (as in most general equilibrium trade literature) using the Lagrange multiplier for the expenditure minimization, then we will have the explicit utility $U = Y/P$, where Y is the national income, and P the price index, which is not the case for the CDE.

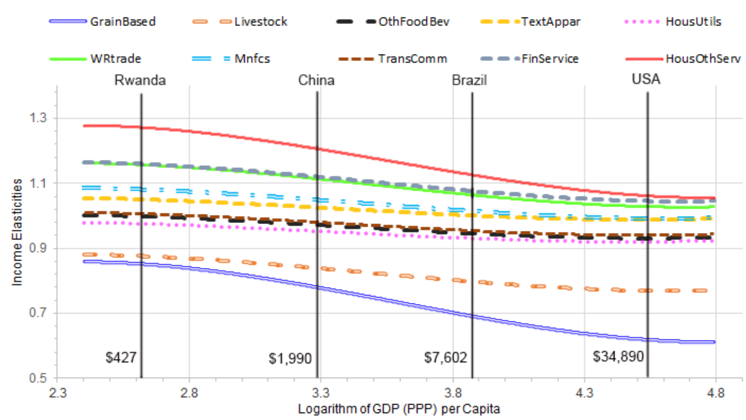

FIGURE 2.— Evolution of Income Elasticities with Real per Capita GDP (2011) (3rd Order Polynomial Trendline) (not including real per capita expenditures that are not categorized into the GTAP commodities).

demand relationship, the constant difference of elasticity or CDE, directly in a maximum likelihood framework. In doing this, we argue that its global regularity conditions stated in Hanoch (1975) can be slightly relaxed, and that the parametric conditions facilitate estimation. We introduce a normalization scheme that is beneficial for the scaling of the parameter values and which appears to have little impact on the economic performance of the estimated system.

We contribute to the literature by directly estimating this demand system for the first time. The central finding in our work is that the direct estimation of this type of demand system is tractable and practical. While critics may object to the fact that we estimate the unobservable cardinal value of utilities, we argue that we do so no more than those who estimate standard CES functions. That is, econometricians estimate all of the parameters necessary to *evaluate* utility, and so may as well have estimated utility. Because the system we estimate is implicit, we have no choice but to explicitly estimate utility. Finally, we identify subtle and important parametric relationships that inspire normalization procedures that achieve identification and result in parameter estimates that are generally well-scaled. We find that the parametric restrictions that result in an identified set of parameters with good scaling are $G(\mathbf{p}/w, u) \equiv \sum_k \beta_k \equiv 1$ and $\sum_k e_k \equiv N$, where N is the number of goods.

On the investigation of the robustness of our parameter estimates, we use a series of numerical tests to verify that the parameter values cannot be changed by an economically significant amount without reducing the likelihood function, suggesting that additional normalizations are not needed for parameter identification. Thus our estimation procedure appears to be computationally tractable and the parameter values identified in the context of a direct maximum likelihood estimation of the Constant Difference of Elasticity preference relationship.

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APPENDIX A: THE PROOF OF MONOTONICITY OF THE CRES

PROOF: For the convenience of the reader we restate the CRES function here:

$$\mathcal{H}(\mathbf{x}, u) = \sum_k \beta_k u^{-e_k(1-\alpha_k)}(x_k)^{1-\alpha_k} \equiv 1, \tag{A.1}$$

where (i) $\beta_k, e_k > 0 \ \forall \ k \in N$, and (ii) either $\alpha_k > 1$ or $0 \leq \alpha_k < 1 \ \forall \ k \in N$. (Note that this argument can be patched to handle $0 \leq \alpha_k < 1$ by dealing with cases. However, the fact that, when we interpret $x_k^{1-\alpha_k} = \ln(x_k)$ when $\alpha_k = 1$, the derivative of $x_k^{1-\alpha_k}$ with respect to x_k remains positive, means that the argument below goes through.)

Take the total differential to get:

$$-\sum_k \beta_k e_k (1-\alpha_k) u^{-e_k(1-\alpha_k)-1} x_k^{1-\alpha_k} du + \sum_k \beta_k u^{-e_k(1-\alpha_k)(1-\alpha_K) x_k^{-\alpha_k} dx_k} \equiv 0. \tag{A.2}$$

Now let all $dx_k = 0$ except for $k = m$ to derive:

$$-\sum_k \beta_k e_k (1-\alpha_k) u^{-e_k(1-\alpha_k)-1} x_k^{1-\alpha_k} du + \beta_m u^{-e_m(1-\alpha_m)} (1-\alpha_m) x_m^{-\alpha_m} dx_m \equiv 0. \tag{A.3}$$

Then we solve this for the change in u for a change in x_m:

$$\frac{du}{dx_m} = \frac{\beta_m u^{-e_m(1-\alpha_m)} (1-\alpha_m) x_m^{-\alpha_m}}{\sum_k \beta_k e_k (1-\alpha_k) u^{-e_k(1-\alpha_k)-1} x_k^{1-\alpha_k}}. \tag{A.4}$$

Notice that the numerator has the sign of $(1-\alpha_m)$, and every term in the sum in the denominator has the sign of $(1-\alpha_k)$. These signs are the same by the parametric restrictions, and hence:

$$\frac{du}{dx_m} > 0. \tag{A.5}$$

Thus, u is strictly monotonic in each x_k. *Q.E.D.*