A Note on the CES Functional Form and Its Use in the GTAP Model

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GTAP Research Memorandum No. 2

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Table of Contents

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Introduction

The purpose of this note is to provide an exhaustive reference for those interested in learning more about the Constant Elasticity of Substitution (CES) function and its use in the representation of producer behavior in the GTAP model.[1] Particular attention is paid to the role of technical change variables and their effect on cost minimizing demands and input shares.

This note is divided into three sections. In the first section, the basic cost minimization problem is laid out and conditional factor demands, as well as the unit cost function, are derived. In section two, this system of equations is expressed in terms of proportional changes, as currently specified in GTAP. This greatly facilitates decomposition of predicted changes in demands and costs between three effects, namely expansion, substitution, and technical change effects. Section two also shows the relationship between changes in cost shares and changes in prices and factor-biased technical change variables. Finally, section three relates these derivations to the notation employed in GTAP.

[1] The CES production function was first developed by: Arrow, K.J, Chenery, H.B., Minhas, B.S., and Solow, R.M. (1961). Capital Labor Substitution and Economic Efficiency. *Review of Economics and Statistics*, 43, pp. 225-250

I. The Producer Cost Minimization Problem

Consider the following producer cost minimization program, where we assume for simplicity a two input production function:

$$\begin{aligned} &\underset{x_1,x_2}{Min} \quad p_1x_1 + p_2x_2 \\ &s.t. \quad y = a\left(d_1x_1^{-r} + d_2x_2^{-r}\right)^{-1/r} \end{aligned} \tag{1}$$

with the following notation:

- x_1 and x_2 are the input levels, p_1 and p_2 their market prices,
- y the production level,
- a the efficiency parameter, with $a > 0$
- d_1 and d_2 are distribution parameters, and
- $s = 1/(1+r)$ the constant elasticity of substitution, with $r > -1$.

The form of the CES production function used in (1) is the one defined in many economic textbooks.[2] This production function satisfies the following properties: it is defined for positive levels of inputs, continuous, differentiable, monotonic and strictly quasi-concave. Moreover, this special form exhibits constant returns to scale. The latter characteristic is commonly adopted in CGE models with perfect competition, like the standard version of GTAP. Accordingly, we will assume constant returns to scale in all of what follows.

The first-order conditions for an interior solution are given by:

$$\frac{p_1}{p_2} = \frac{d_1x_1^{-r-1}}{d_2x_2^{-r-1}} \tag{2}$$

or

$$x_2 = x_1\left(\frac{d_2p_1}{d_1p_2}\right)^{1/(1+r)} \tag{3}$$

Now substituting equation (3) back into the CES production function, we obtain:

$$\left(\frac{y}{a}\right)^{-r} = d_1x_1^{-r} + d_2x_1^{-r}\left(\frac{d_2p_1}{d_1p_2}\right)^{-r/(1+r)} \tag{4}$$

[2]There are other forms of the CES that differ in terms of specification of distribution parameters. For another widely used representation (called the calibrated share form) of the CES, see Rutherford T. (1995). Constant Elasticity of Substitution Functions: Some Hints and Useful Formulae. Available at http://www1.gams.com/solvers/mpsge/cesfun.htm.

Gathering terms:

$$\left(\frac{y}{a}\right)^{-r} = x_1^{-r}\left(\frac{p_1}{d_1}\right)^{-r/(1+r)}\left(d_1^{1-r/(1+r)} p_1^{r/(1+r)} + d_2^{1-r/(1+r)} p_2^{r/(1+r)}\right) \tag{5}$$

Isolating x_1, we have the conditional demand for the first input:

$$x_1 = \frac{y}{a}\left(\frac{d_1}{p_1}\right)^{1/(1+r)}\left(d_1^{1/(1+r)} p_1^{r/(1+r)} + d_2^{1/(1+r)} p_2^{r/(1+r)}\right)^{1/r} \tag{6}$$

Plugging (6) into (3) yields the conditional demand for the second input:

$$x_2 = \frac{y}{a}\left(\frac{d_2}{p_2}\right)^{1/(1+r)}\left(d_1^{1/(1+r)} p_1^{r/(1+r)} + d_2^{1/(1+r)} p_2^{r/(1+r)}\right)^{1/r} \tag{7}$$

These conditional demands display the following properties: non-decreasing and homogenous of degree one with respect to the level of production, homogenous of degree zero with respect to input prices, non-increasing with respect to the own-price and non-decreasing with respect to the other input price.

Owing to the constant returns to scale property of our CES production function, we can further simplify these conditional demand equations, by introducing the unit cost c_y, which we will define as follows:

$$c_y = C(y, p_1, p_2)/y = (p_1 x_1 + p_2 x_2)/y \tag{8}$$

Substituting (6) and (7) into (8) leads the following unit cost function:

$$c_y(p_1, p_2) = (p_1 x_1 + p_2 x_2)/y = \frac{1}{a}\left(d_1^{1/(1+r)} p_1^{r/(1+r)} + d_2^{1/(1+r)} p_2^{r/(1+r)}\right)^{1+1/r} \tag{9}$$

This unit cost function respects the desired properties, including: non-decreasing, concave and continuous with respect to each input price and homogenous of degree one with respect to the input prices.

Note that unit cost is invariant to output level so that the total cost function is homogenous of degree one with respect to the production level. (This follows from the constant returns to scale assumption.) Therefore, unit cost also equals marginal cost, which is also independent of the level of production. We can now express conditional demands as a function of unit cost by rearranging (9) as follows:

$$A = \left(a\ c_y\right)^{r/1+r} \quad (10)$$

where $$A = \left(d_1^{1/1+r}\, p_1^{r/1+r} + d_2^{1/1+r}\, p_2^{r/1+r}\right) \quad (11)$$

Substituting this expression for (10) into the conditional demand equations (6) and (7), we can rewrite them as a function of the marginal cost:

$$x_1 = \frac{y}{a}\left(\frac{d_1}{p_1}\right)^{1/1+r}\left(\left(a\ c_y\right)^{r/1+r}\right)^{1/r} \quad (12)$$

This can be simplified as follows:

$$x_1 = y\left(\frac{d_1\ c_y}{p_1}\right)^{s}\ a^{\,s-1} \quad (13)$$

In the same way, we have:

$$x_2 = y\left(\frac{d_2\ c_y}{p_2}\right)^{s}\ a^{\,s-1} \quad (14)$$

Equations (13) and (14) show that the impact of a change in relative prices on conditional demands depends on the change in unit cost relative to the own price of the input. If $\frac{c_y}{p_i}$ rises, then more of the input will be demanded, for any given level of output.

II. Linearization of Conditional Demands and Costs Functions

II.A. Mathematical preliminaries

In order to linearize the CES demand equations, we make use of the following rules of linearization, where the hat ^ denotes proportional changes ($\hat{A} = dA/A$). The reader can verify these rules as her own using the basic rules of differentiation.

$$\widehat{(AB)} = \hat{A} + \hat{B}$$

$$\widehat{(A/B)} = \hat{A} - \hat{B}$$

$$\left(\hat{B^s}\right) = s \ \hat{B}$$

$$(A \hat{+} B) = \frac{A}{A+B}\hat{A} + \frac{B}{A+B}\hat{B}$$

II.B. Linearization of conditional demands

Using the above linearization rules, the conditional demand equations (13) and (14) become:

$$\hat{x}_1 = \hat{y} + s\left(\hat{d}_1 + \hat{c}_y - \hat{p}_1\right) + (s-1)\,\hat{a} \qquad (15)$$

$$\hat{x}_2 = \hat{y} + s\left(\hat{d}_2 + \hat{c}_y - \hat{p}_2\right) + (s-1)\,\hat{a} \qquad (16)$$

These equations offer a natural decomposition of changes in derived demands into those attributable to:

* expansion effect = $\hat{y}$, i.e. the effect of a change in output level,
* substitution effect = $s\left(\hat{c}_y - \hat{p}_i\right)$, i.e. the effect of a change in relative prices,
* biased technical change = $s\ \hat{d}_i$, i.e. the effect of factor biased technical change,
* neutral technical change = $(s-1)\,\hat{a}$, i.e. the effect of neutral technical change on demand.

II.C. Linearization of the unit cost function

There are two ways to linearize the unit cost function. The first approach to linearization is the “brute force” method. The second takes advantage of the fact that the unit cost function is homogeneous of degree one in prices.

II.C.1. First approach

From equation (10), we have:

$$a\ c_y = \left(d_1^{1/(1+r)} p_1^{r/(1+r)} + d_2^{1/(1+r)} p_2^{r/(1+r)}\right)^{(r+1)/r} = A^{(r+1)/r} \qquad (17)$$

Complete differentiation of equation (17) yields:

$$\alpha \; dc_y + c_y \; d\alpha \; = \frac{r+1}{r} \, A^{-1+{r+1}/{r}}$$

$$\left(\begin{array}{l} \frac{1}{1+r} d_1^{-1+1/(1+r)} \; p_1^{r/(1+r)} \; dd_1 + \frac{r}{1+r} d_1^{1/(1+r)} \; p_1^{-1+r/(1+r)} \; dp_1 \\ + \; \frac{1}{1+r} d_2^{-1+1/(1+r)} \; p_2^{r/(1+r)} \; dd_2 + \frac{r}{1+r} d_2^{1/(1+r)} \; p_2^{-1+r/(1+r)} \; dp_2 \end{array} \right) \tag{18}$$

This can be simplified as follows:

$$\alpha \; c_y \left(\hat{\alpha} + \hat{c}_y \right) = \; A^{-1+{r+1}/{r}} \left(\begin{array}{l} d_1^{1/(1+r)} p_1^{r/(1+r)} \left(\frac{\hat{d}_1}{r} + \hat{p}_1 \right) \\ + d_2^{1/(1+r)} p_2^{r/(1+r)} \left(\frac{\hat{d}_2}{r} + \hat{p}_2 \right) \end{array} \right) \tag{19}$$

$$\begin{aligned} \hat{c}_y + \hat{\alpha} \; = \; & \frac{d_1^{1/(1+r)} p_1^{r/(1+r)}}{d_1^{1/(1+r)} p_1^{r/(1+r)} + d_2^{1/(1+r)} p_2^{r/(1+r)}} \left(\frac{\hat{d}_1}{r} + \hat{p}_1 \right) \\ + \; & \frac{d_2^{1/(1+r)} p_2^{r/(1+r)}}{d_1^{1/(1+r)} p_1^{r/(1+r)} + d_2^{1/(1+r)} p_2^{r/(1+r)}} \left(\frac{\hat{d}_2}{r} + \hat{p}_2 \right) \end{aligned} \tag{20}$$

Equation (20) can be further simplified by recalling equation (3), which permits us to write:

$$\frac{x_2}{x_1} = \frac{d_2^{1/(1+r)} p_2^{-1/(1+r)}}{d_1^{1/(1+r)} p_1^{-1/(1+r)}} \tag{21}$$

Multiplying equation (21) by the price ratio, we obtain

$$\frac{p_2 x_2}{p_1 x_1} = \frac{d_2^{1/(1+r)} p_2^{r/(1+r)}}{d_1^{1/(1+r)} p_1^{r/(1+r)}} \tag{22}$$

and $$\frac{p_2 x_2}{p_1 x_1} + 1 = \frac{p_2 x_2 + p_1 x_1}{p_1 x_1} = \frac{d_2^{1/(1+r)} p_2^{r/(1+r)} + d_1^{1/(1+r)} p_1^{r/(1+r)}}{d_1^{1/(1+r)} p_1^{r/(1+r)}} \tag{23}$$

Therefore, equation (20) can be written as:

$$\hat{c}_y + \hat{a} = \frac{p_1 x_1}{p_1 x_1 + p_2 x_2}\left(\frac{\hat{d}_1}{r} + \hat{p}_1\right) + \frac{p_2 x_2}{p_1 x_1 + p_2 x_2}\left(\frac{\hat{d}_2}{r} + \hat{p}_2\right) \quad (24)$$

Now let us adopt the following notation for cost shares:

$$q_i = \frac{p_i x_i}{p_1 x_1 + p_2 x_2} = \frac{p_i x_i}{c_y y}, i = 1,2 \quad (25)$$

We can finally write:

$$\hat{c}_y + \hat{a} = q_1\left(\frac{\hat{d}_1}{r} + \hat{p}_1\right) + q_2\left(\frac{\hat{d}_2}{r} + \hat{p}_2\right) \quad (26)$$

Thus the change in unit cost is a function of the three different types of technical change, as well as the two prices.

II.C.2. Second approach

In the second approach, we take advantage of the Euler theorem and Shephard's lemma. Start with the unit cost function (10). We can rewrite it in the following way:

$$c_y^* = \left(\left(p_1^*\right)^{r/1+r} + \left(p_2^*\right)^{r/1+r}\right)^{r+1/r} \quad (27)$$

with $c_y^* = a\ c_y$, $p_i^* = d_i^{1/r}\ p_i$. Then we can write:

$$\hat{c}_y^* = e_{p_1^*}^{c_y^*}\ \hat{p}_1^* + e_{p_2^*}^{c_y^*}\ \hat{p}_2^* = e_{p_1^*}^{c_y^*}\left(\frac{\hat{d}_1}{r} + \hat{p}_1\right) + e_{p_2^*}^{c_y^*}\left(\frac{\hat{d}_2}{r} + \hat{p}_2\right) \quad (28)$$

Our objective is then to derive expressions for elasticities included in expression (28). In this respect, first we make use of the fact that the unit cost function is linearly homogenous with respect to input prices. Applying Euler's theorem, we obtain:

$$c_y^* = \frac{\P c_y^*}{\P p_1^*}\ p_1^* + \frac{\P c_y^*}{\P p_2^*}\ p_2^* \quad (29)$$

Dividing (29) by the left hand side, we have:

$$1 = e_{p_1^*}^{c_y^*} + e_{p_2^*}^{c_y^*} \quad (30)$$

We therefore know that the sum of these elasticities equals one. Now, using Shephard's lemma, we can write:

$$e^{c_y^*}_{p_1^*} = \frac{\P\left(a\ c_y\right)}{\P\left(d_1^{1/r}p_1\right)}\frac{d_1^{1/r}p_1}{a\ c_y} = \frac{\P\ c_y}{\P p_1}\frac{1}{\frac{\P\left(d_1^{1/r}p_1\right)}{\P p_1}}\frac{d_1^{1/r}p_1}{c_y} \tag{31}$$

$$e^{c_y^*}_{p_1^*} = \frac{\P\ c_y}{\P\ p_1}\frac{p_1}{c_y} = \frac{p_1 x_1}{c_y y} = q_1 \tag{32}$$

Substitution of these cost shares into (28) reproduces the same expression for the change in unit costs that is provided in (26).

II.D. Evolution of cost shares under technical change effects

At this point, we are in a good position to analyze the effects of technical change on the evolution of cost shares. For example, differentiation of the cost share for input 1 (equation 25) gives:

$$\hat{q}_1 = \hat{p}_1 + \hat{x}_1 - \hat{c}_y - \hat{y} \tag{33}$$

Using equations (15) and (26), we obtain:

$$\hat{q}_1 = \hat{p}_1 + \hat{y} + s\left(\hat{d}_1 + \hat{c}_y - \hat{p}_1\right) + (s-1)\hat{a} - \hat{c}_y - \hat{y} \tag{34}$$

$$\hat{q}_1 = \hat{p}_1(1-s) + \hat{d}_1 s + (s-1)\left(q_1\left(\frac{\hat{d}_1}{r} + \hat{p}_1\right) + q_2\left(\frac{\hat{d}_2}{r} + \hat{p}_2\right)\right) \tag{35}$$

$$\hat{q}_1 = (1-s)\left(\hat{p}_1 + \frac{\hat{d}_1}{r}\right) + (s-1)\left(q_1\left(\hat{p}_1 + \frac{\hat{d}_1}{r}\right) + q_2\left(\hat{p}_2 + \frac{\hat{d}_2}{r}\right)\right) \tag{36}$$

$$\hat{q}_1 = q_2(1-s)\left(\left(\hat{p}_1 + \frac{\hat{d}_1}{r}\right) - \left(\hat{p}_2 + \frac{\hat{d}_2}{r}\right)\right) \tag{37}$$

Now, define

$$\hat{p}_{ei} = \hat{p}_i + \frac{\hat{d}_i}{r}$$

as the proportional change in the "effective" price of input i. Then the proportional change in the cost share of input 1 can be expressed as:

$$\hat{q}_1 = q_2 (1 - s) \left(\hat{p}_{e1} - \hat{p}_{e2} \right) \tag{38}$$

This expression for the change in the cost share is quite informative. In particular, we note the following important propositions:

P1) The rate of change in the cost share of input 1 is directly related to the size of the cost share of input 2. In the extreme, if $q_2 = 0$ and $q_1 = 1$, then $\hat{q}_1 = \hat{q}_2 = 0$. As the importance of input 2 increases, the responsiveness of the cost share of input 1 to changes in the effective prices increases.

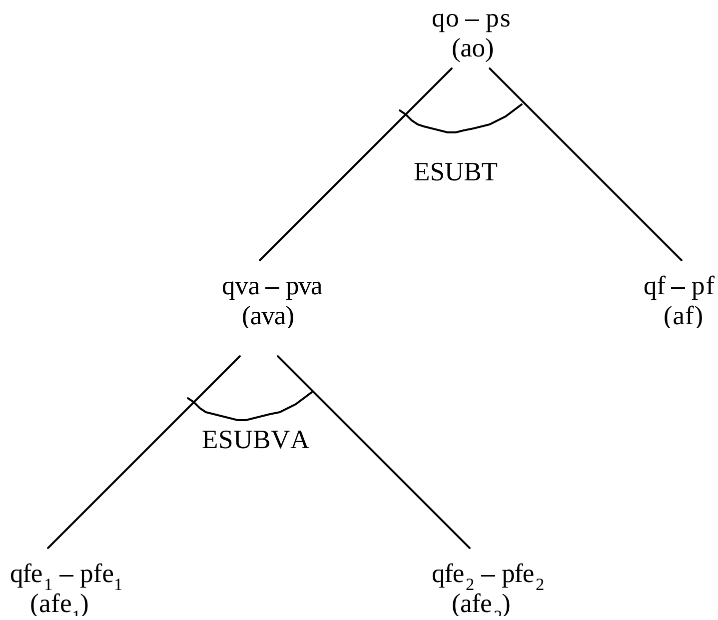
P2) In the Cobb-Douglas case, $s = 1$, we necessarily have $\hat{q}_1 = \hat{q}_2 = 0$. The cost shares are invariant to both price changes, as well as technical changes. In this special production function, the shares are parameters.

P3) The relationship between $\hat{q}_1$ and the change in the ratio of effective prices

$\widehat{\left(\frac{p_{e1}}{p_{e2}} \right)} = \hat{p}_{e1} - \hat{p}_{e2}$, depends on whether s is greater than or less than 1. When $s > 1$, the cost share of input 1 is decreasing in its relative effective price, whereas when $s < 1$ the opposite is true.

III. Correspondence with GTAP Notation

The CES functional form is used in the standard GTAP model at different stages. We focus here on the producer level. The reader can work out the implications of other CES representations in an analogous fashion. At the producer level, the CES is used to specify the substitution possibilities between intermediate inputs and value-added as well as between primary factors in the value-added nested. The GTAP notation used at this level is summarized with the following production tree that links quantities and prices, as well as showing where the technical change terms enter. To simplify the presentation, we will assume that there are two primary factors of production and only one intermediate input. Extensions to many intermediate inputs and primary factors of production is straightforward.



III.A. The value-added nest

We first consider the value-added nest, where only biased technical change terms are specified. Producers combine primary factors of production, according to a CES function with substitution elasticity ESUBVA. In order to utilize the equations from section II, let define the following correspondence:

$$qfe_i = \hat{x}_i\,, \qquad pfe_i = \hat{p}_i$$

$$qva = \hat{y}\,, \qquad pva = \hat{c}_y$$

$$afe_i = \frac{s}{s-1}\hat{d}_i = -\frac{\hat{d}_i}{r}\,, \qquad ESUBVA = s$$

$$SVA_i = q_i\,, \qquad \hat{a} = 0$$

Using equations (15) or (16), the conditional demand of primary factors of production are given by:

$$qfe_i = qva + ESUBVA\left(\frac{ESUBVA-1}{ESUBVA}afe_i + pva - pfe_i\right) \tag{39}$$

Rearranging yields the equation ENDWDEMAND of the GTAP TAB file:

$$qfe_i = -afe_i + qva - ESUBVA\left(pfe_i - afe_i - pva\right) \tag{40}$$

The unit cost of value added is obtained from equation (26):

$$pva = \sum_i SVA_i\left(pfe_i - afe_i\right) \tag{41}$$

This corresponds to the equation labeled VAPRICE in the GTAP TAB file.

III.B. The top level of the production tree

At the top level of the production tree, producers combine value-added and intermediate inputs, according to a single CES function with substitution elasticity ESUBT. Factor biased and neutral technical change terms are specified at this stage. Following the procedure used above, define the following correspondence:

$$qf = \hat{x}_1, \qquad pf = \hat{p}_1$$

$$qva = \hat{x}_2, \qquad pva = \hat{p}_2$$

$$qo = \hat{y}, \qquad ps = \hat{c}_y$$

$$af = \frac{s}{s-1}\hat{d}_1 = -\frac{\hat{d}_1}{r}, \qquad ava = \frac{s}{s-1}\hat{d}_2 = -\frac{\hat{d}_2}{r}$$

$$ao = \hat{a}, \qquad ESUBT = s$$

$$q_f = q_1, \qquad q_{va} = q_2$$

Again, using equations (15) and (16), the conditional demand of the value added and the intermediate input are given by:

$$qva = qo + ESUBT\left(\frac{ESUBT-1}{ESUBT}ava + ps - pva\right) + (ESUBT-1)\,ao \tag{42}$$

$$qf = qo + ESUBT\left(\frac{ESUBT-1}{ESUBT}af + ps - pf\right) + (ESUBT-1)\,ao \tag{43}$$

Rearranging yields the equations VADEMAND and INTDEMAND of the GTAP TAB file:

$$qva = -ava + qo - ao - ESUBT\left(pva - ava - ps - ao\right) \tag{44}$$

$$qf = -af + qo - ao - ESUBT\left(pf - af - ps - ao\right) \tag{45}$$

The unit cost of the output is also obtained from equation (26):

$$ps + ao = q_f\left(pf - af\right) + q_{va}\left(pva - ava\right) \tag{46}$$

The latter corresponds to the zero profit condition in GTAP. In the GTAP TAB file, the price of value added is replaced by its expression obtained in equation (41).