Necessary and unnecessary parameter restrictions for CDES demand systems

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Abstract

In a 1975 *Econometrica* paper, Giora Hanoch proposed the so-called CDES functional form of a consumer demand system. Though not very popular in econometric applications, this functional form is a good choice for modelling consumer demand in CGE analysis, because it is a good compromise between the desire for flexibility on the one hand and parametric parsimony on the other. It allows to choose own-price and income elasticities subject to the restriction that the so-called expansion parameters must all be nonnegative and not all zero, and that the so-called substitution parameters must not exceed one. If one believed in the literature, one furthermore would have to impose the restriction that the substitution parameters must all lie on the same side of zero. The main purpose of this paper is to show that this restriction is unnecessary. Any sign pattern of substitution parameters can be admitted without destroying global regularity of the demand system. For proving this result, the paper defines the CDES system in a form differing slightly from the form suggested by Hanoch; instead of a sum of power functions the implicit form is defined as a sum of Box-Cox transforms. The paper covers a full analysis of the properties of the function; it discusses the flexibility of the form showing that even without the unnecessary restriction it is still not flexible enough to reproduce any desired regular pattern of own-price and income elasticities. An optimisation approach to parameter calibration subject to the required sign constraints may still be needed. It is demonstrated that calibrations with and without the mentioned unnecessary restriction can lead to entirely different parameters and different behaviour of the calibrated function. Differences between calibrations with and without the unnecessary restriction are illustrated using the numerical example in Huff et al. (1997). The final Section of the paper shows that the form presented in this paper is (almost) equivalent to the original Hanoch form. “Almost” means that, with measure zero in parameter space, data can be such that they allow a representation in our, but not in Hanoch’s form. Apart from this “non-generic” case, the original Hanoch form and our form with a suitable parameter transformation define the same demand system.

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1 Introduction

In the last century mathematical economists were eager for functional forms of demand system that allowed for keeping certain elasticities unchanged while prices and income vary. This seemed attractive because elasticities are dimensionless parameters, and keeping them constant would allow to regard them as some kind of constants of nature that can be applied to any concrete environment, once they are estimated with sufficient precision. Unfortunately, authors such as Uzawa, McFadden, Gorman and others found that the requirement of regularity of the demand systems, combined with constant elasticities, led to functions too inflexible to be able to reproduce important characteristics of observed demand behaviour (Gorman; 1965). Hence, in the next round of searching for functional forms authors asked whether one can allow for sufficient flexibility by weakening the required constancy of elasticities. Instead of wanting them to be constant one tried to allow them to vary in a simple way. Gorman (1965) and others looked for a form where the so-called Allen-Uzawa elasticities of substitution (AUES) — though varying with varying exogenous variables — stood in constant ratios to each other. In a seminal paper Hanoch (1971) succeeded in working the idea out. He presented a homothetic version of such a function and fully elaborated its formal properties. The form was baptised CRESH for constant ratio elasticity of substitution, homothetic. In particular, this publication was the first (and apparently also the last one dealing with the issue, see below) that exhaustively described the parameter restrictions required to render this form globally regular, i.e. to equip it with the monotonicity, homotheticity and curvature properties that the function should have. Hanoch carried the subject on in a second *Econometrica* paper (Hanoch; 1975) by showing first how non-homotheticity can be allowed for (the CRES form), and second, by proposing to use the same functional form in (normalised) price space rather than quantity space. While CRES defines in an implicit form the relation between utility and consumption, the application to the price space defines a relation between utility and normalised prices. Normalized prices are prices divided by expenditure. It turned out that this concept generated a demand system with AUES coming in constant differences rather than ratios. Consequently it was called CDES (constant difference elasticity of substitution).

The usual way to define the CDES, called the *Hanoch-form* in the following, is

\[ \sum_i B_i u^{\alpha_i \delta_i} y_i^{\delta_i} = 1, \]  

with variables \( u \) and \( y_i \). \( u \) denotes utility and \( y_i \) denotes the normalized price of commodity \( i \), i.e. the price \( p_i \) divided by expenditure \( m \). With \( n \) commodities there are \( 3n \) parameters, the distribution parameters \( B_i \), the expansion parameters \( \alpha_i \), and the substitution parameters \( \delta_i \). Under suitable parameter restrictions that are to be discussed in detail below, identity (1) defines any of the endogenous variables as functions of the others. Solving for \( u \) gives the indirect utility as a function of \( y := (y_1, \ldots, y_n) \), solving for \( m \) gives the expenditure as a function of \( u \) and \( p := (p_1, \ldots, p_n) \), and solving for \( p_i \) gives the price compensation, which is the maximal price the consumer is willing to pay for commodity \( i \), given other prices, expenditure \( m \), and minimal utility \( u \) to be attained. If \( i \) stands for land, urban economists call this the “bid-price” function for land.

The CDES form never became really popular in econometric work, because more flexible forms have been invented and richer data sets make the estimation of these forms
possible. CDES survived however in CGE analysis, and did so for good reasons. For the large number of commodities distinguished in today’s CGE designs the information for fixing the parameters of fully flexible forms is not available. Restricting the parameter space is unavoidable, and CDES is a comfortable way to do so (Hertel et al.; 1991).

Unfortunately, a fault from the literature creeped in, as the CDES was adapted to CGE analysis, leading to an unnecessary parameter restriction. It is taken for granted in all applications I am aware of that the substitution parameters $\delta_i$ must all have the same sign (either all positive or all negative). As a prominent example see the application in GTAP (Huff et al.; 1997). This restriction is in fact stated in Hanoch’s 1975 paper and has been reiterated ever since, but its necessity has never been tried to prove. In his 1975 paper, Hanoch refers to his 1971 paper for proofs, but in the 1971 paper the mentioned restrictions are absent!

The main task of this paper is to show that the sign restrictions on the substitution parameters can be removed without destroying global regularity, and thereby to help users of the form to fully exploit the flexibility inherent in the CDES, even if this flexibility is still limited in comparison to other less parsimonious forms. To this end, we first present the CDES in a slightly different form than used in the literature, for three reasons (Section 2): first, regularity is immediate from this form; second, the form is most suitable for application in CGE, because benchmark data directly appear as parameters. No extra step is required for computing distribution parameters; and third, the special case $\delta_i = 0$ is included and does not call for an extra treatment.

Then we derive compensated and uncompensated price elasticities, AUES, and income elasticities. The formulae are known, but repeated here for having them handy to invert them, which means to show how the expansion and substitution parameters are derived from price and income elasticities (Section 3). In Section 4 I evaluate the flexibility of the CDES form as presented here and discuss the gain in flexibility over the restricted version of the literature. Section 5 then shows how our version is related to the original Hanoch-form in equation (1), with parameter restrictions as given in the original 1971 paper. I demonstrate almost equivalence in a sense to be explained. Section 6 concludes.

2 Another way of defining the CDES form

We call a consumer demand system regular, if it is generated (or can be conceived to be generated) by maximizing utility subject to a budget constraint, taking prices as given, with utility representing a locally non satiated strictly convex preference relation. A demand system is well known (Mas-Colell et al.; 1995, Ch. 3H) to be regular, if and only if the corresponding expenditure function $e$ is strictly increasing in utility $u$ and, as a function of the price vector $p$, given $u$, has the properties for $p$:

e.1: nonincreasing in all $p_i$,

e.2: homogeneous of degree one,

e.3: concave,

e.4: differentiable.
This in turn is the case if and only if the indirect utility function \( v \) with income \( m \) has the properties for \( p \gg 0^1 \)

v.1: increasing in \( m \) and non-increasing in all \( p_i \),

v.2: homogeneous of degree zero,

v.3: quasiconvex,

v.4: differentiable.

Both, \( e \) and \( v \) contain all information about demand for \( p \gg 0 \), and demand is unique under the stated assumptions.

If we weakened the assumption of strict convexity of preferences by just assuming convexity, everything would still go through, except that we have to dispense with differentiability of \( e \) and \( v \) and with uniqueness of demand. If we furthermore also go without convexity of preferences, then the equivalence still holds, but the two functions do not longer contain the full information because demand vectors may be proper subsets of subdifferentials, and the functions do not tell us which subsets. Differently put, \( e \) and \( v \) do not tell us anything about inward bulges of preferences, if they are admitted. We mention this in order to stress that we do not get rid of the regularity requirements for \( e \) and \( v \) by allowing for less restrictive preferences. It is the rationality of price takers that implies the restrictions.

We prefer writing the indirect utility in normalized prices as follows: Due to property 2 of \( v \) we know that \( v(p,m) = v(p/m,1) =: w(y) \). Hence, a demand system is regular if and only if \( w(y) \), \( w : \mathbb{R}^n \to \mathbb{R} \), has the properties for \( y \gg 0 \):

w.1: negative monotone,\(^2\)

w.2: quasiconvex,

w.3: differentiable.

\( w \) can be expressed in implicit form as

\[
  w(y) = \{ u : F(y,u) = 0 \},
\]

provided \( F \) is either increasing or decreasing in \( u \). Let, for \( u \gg 0 \) and \( y \gg 0 \) the function \( F \) be

F.1: strongly positive monotone,\(^3\)

F.2: concave in \( y \),

F.3: differentiable.

\(^1\)\( p \gg 0 \) means \( p_i > 0 \) for all \( i \).

\(^2\)A function \( f : \mathbb{R}^n \to \mathbb{R} \) is called positive (negative) monotone, if \( z \gg x \) implies \( f(z) > f(x) \) (\( f(z) < f(x) \)).
Assume furthermore, that for all \( y \gg 0 \) there is some \( u > 0 \) such that \( F(y, u) = 0 \). Hence, \( w \) is well defined and positive for all \( y \gg 0 \). The implicit \( w \) then fulfills properties w.1 to w.3 of a regular indirect utility. As to property w.1, \( w \) is obviously strongly negative monotone, hence the weaker property w.1 holds in particular. As to property w.2, the level sets are

\[
\mathcal{L}_u = \{ y : w(y) \leq u \} = \{ y : F(y, w(y)) \leq F(y, u) \} = \{ y : 0 \leq F(y, u) \}.
\]

The first equality holds because \( F \) is increasing in utility, the second is due to \( F(y, w(y)) = 0 \) by definition of \( w \). Due to concavity of \( F \) in \( y \) this is a convex set (Rockafellar; 1970, Th. 4.6). Hence, \( w \) is quasiconvex. Differentiability is obvious.

We are ready now to introduce the CDES demand system by specifying \( F \) as an additive function as follows:

\[
\sum_i s^0_i f(u^{\alpha_i}y_i/y^0_i, \delta_i) = 0, \tag{2}
\]

with \( f : \mathbb{R}^2 \to \mathbb{R} \) being the Box-Cox transform:

\[
f(x, \delta) := \begin{cases} 
  x^{\delta-1} & \text{for } \delta \neq 0 \\
  \log x & \text{for } \delta = 0.
\end{cases}
\]

The definition for \( \delta = 0 \) is natural since \( \lim_{\delta \to 0} f(x, \delta) = \log x \) by l’Hospital’s rule. \( s^0_i > 0 \) with \( \sum_i s^0_i = 1 \), \( \alpha_i \) with \( \alpha_i \geq 0 \) for all \( i \) and \( \alpha_i > 0 \) for at least one \( i \), \( y^0_i > 0 \) and \( \delta_i \leq 1 \) are parameters.

The function \( F(y, u) \) defined by the left-hand side of equation (2) is clearly concave in \( y \) and differentiable, because \( f \) has these properties for all \( i \) and they are preserved under positive summation. \( f \) has the derivative \( f' = x^{\delta-1} \). It exists for \( x > 0 \), is positive and non-increasing for \( \delta \leq 1 \). Hence, the function is differentiable, increasing and concave. Strong monotonicity is also obvious: partial derivatives with respect to all \( y_i \) are positive, and the derivative with respect to \( u \) is positive as well, because it is non-negative in the terms for all \( i \) and positive in at least one term.

Specifying the CDES in this form has not only the advantage of including the possible case \( \delta = 0 \) without special treatment, it also shows very clearly that restrictions on the substitution parameters other then \( \delta_i \leq 1 \) are not required. Another advantage is that this form is most suitable for calibrating the function for CGE applications. We therefore call it the calibrated form of CDES. As specified, the form has \( 4n \) parameters, unlike the original form (1) that needs only \( 3n \). Hence, the parametric form is not parsimonious in the sense of Lau (1984). This superficial disadvantage turns out to be an advantage, however, because the parameters \( s^0_i \) and \( y^0_i \) are directly inserted from benchmark data, if the function is calibrated.

Let us, for the time being, assume that \( \alpha_i \) and \( \delta_i \) were known for all \( i \). Let \( s^0_i \) denote expenditure shares of an observed benchmark, and let \( y^0_i = p^0_i/m^0 \) denote normalized expenditure shares of the observed benchmark. A function \( f : \mathbb{R}^n \to \mathbb{R} \) is called strongly positive (negative) monotone, if \( z \geq x \) and \( z \neq x \) imply \( f(z) > f(x) \) (\( f(z) < f(x) \)). In one dimension, monotonicity and strong monotonicity are the same. In the one-dimensional case we call a positive (negative) monotone function “increasing” (“decreasing”). In general, strong monotonicity implies monotonicity, but not vice versa.
benchmark prices. Often $p_i = 1$ is chosen in practice, such that $y_i/y_i^0 = p_im^0/m$ in this case, but any other choice is allowed. Then for $u^0 = 1$ and $y_i = y_i^0$ equation (2) obviously holds (note that $f(1, \delta) = 0$ for all $\delta$), and the expenditure share generated by (2) is $s_i = s_i^0$. Hence, with $y_i = y_i^0$ the benchmark is reproduced, with benchmark utility level equal to one.

To derive the formula for the expenditure shares, observe that (2) may be read as an implicit expenditure function, if $u$ is held constant. Totally differentiating with constant $u$ gives

$$dp_i \frac{1}{p_i} s_i^0(u^\alpha y_i/y_i^0)^{\delta_i} - dm \frac{1}{m} \sum_j s_j^0(u^\alpha y_j/y_j^0)^{\delta_j} = 0.$$ 

The expenditure share is the elasticity of expenditures with respect to the price, which yields

$$s_i = \frac{dm}{dp_i} = \frac{s_i^0(u^\alpha y_i/y_i^0)^{\delta_i}}{\sum_j s_j^0(u^\alpha y_j/y_j^0)^{\delta_j}}.$$ 

Clearly this implies $s_i = s_i^0$ for $y_i = y_i^0$ and $u = u^0 = 1$. From (3) we obtain the Hicksian and Marshallian demand. The Hicksian demand is $h_i(p, u) = m s_i/p_i$, with $m$ implicitly fixed by (2), given $u$. Similarly, Marshallian demand is $x_i(p, m) = m s_i/p_i$ with $u$ implicitly fixed by (2), given $m$.

For the compensated price elasticity we obtain (see Appendix)

$$\psi_{ij} := \frac{\partial \log h_i(p, u)}{\partial \log p_j} = s_j(\sigma_i + \sigma_j - \bar{\sigma}) - \Delta_{ij}\sigma_i,$$ 

with $\sigma_i := 1 - \delta_i > 0$, $\bar{\sigma} = \sum_i s_i \sigma_i$, and $\Delta_{ij} = 1$ for $i = j$ and $\Delta_{ij} = 0$ else. The AUES is

$$\phi_{ij} := \psi_{ij}/s_j = \sigma_i + \sigma_j - \bar{\sigma} - \Delta_{ij}\sigma_i/s_j,$$ 

the well known formula giving the CDES its name (see Hanoch (1975), Equ. (3.6)). The income elasticity is

$$\eta_i := \frac{\partial \log x_i(p, m)}{\partial \log m} = (\alpha_i + \bar{\alpha}\sigma - \alpha_i\sigma_i)/\bar{\alpha} + \sigma_i - \bar{\sigma},$$ 

with $\bar{\alpha} := \sum_i s_i \alpha_i$ and $\bar{\alpha}\sigma := \sum_i s_i \alpha_i \sigma_i$. The direct price elasticity can be obtained from (5) and (6) by Slutsky’s equation,

$$\epsilon_{ij} := \frac{\partial \log x_i(p, u)}{\partial \log p_j} = s_j(\phi_{ij} - \eta_i).$$

### 3 Calibrating elasticity parameters

If the expansion and substitution parameters are given, the other parameters can always be chosen such that any benchmark observation for the demand and price vector can be reproduced. We just have to insert observed expenditure shares for $s_i^0$ and observed normalized prices for $y_i^0$. But what about the expansion and substitution parameters? They should be calibrated such that observed price and income elasticities are reproduced as close as possible. We therefore call these parameters elasticity parameters. Clearly there are not sufficient parameters for reproducing any regular elasticity pattern, that is
the form is not fully flexible. But one can at least hope to reproduce own-price elasticities and income elasticities. These are \(2n\) observations for \(2n\) parameters. Engel aggregation, requiring \(\sum_i \eta_i s_i = 1\), implies that only \(2n - 1\) of these observations are independent. This corresponds to the fact that a scaling restriction can be imposed on the expansion parameters, so that \(2n - 1\) parameters remain to be calibrated by \(2n - 1\) independent observations.

An obvious procedure is first to solve

\[
\psi^0_i = s^0_i (2\sigma_i - \bar{\sigma}) - \sigma_i
\]

for \(\sigma\), and then

\[
\eta^0_i = (\alpha_i + \bar{\alpha} \sigma - \alpha_i \sigma_i) / \bar{\alpha} + \sigma_i - \bar{\sigma},
\]

for \(\alpha\), given \(\sigma\). A look at (2) shows that scaling \(\alpha\) by an arbitrary positive factor just induces an increasing transformation of utility without affecting the demand system. We thus can restrict \(\alpha\) such that \(\sum_i s^0_i \alpha_i = 1\), simplifying (8) to

\[
\eta^0_i = \alpha_i + (\sigma_i - \bar{\sigma}) - (\alpha_i \sigma_i - \alpha \sigma).
\]

This is more intuitive: the income elasticity equals the expansion parameter, corrected for deviations of \(\sigma_i\) and \(\alpha_i \sigma_i\) from their respective averages. Note, however, that (9) only holds at the benchmark. For prices deviating from the benchmark, \(\bar{\alpha}\) deviates from one, in general.

Both equations (7) and (9) are linear in the respective parameters, making the solution easy, as noted by Hertel et al. (1991). Solutions are most conveniently written in vector notation. Let \(\psi^0\), \(s^0\) and \(\sigma\) denote column vectors with coordinates \(\psi^0_{ii}\), \(s^0_s i\) and \(\sigma_i\), respectively, and let \(\text{diag}(2s - 1)\) denote a diagonal matrix with \(i\)-th diagonal element \(2s_i - 1\), then

\[
\sigma = A^{-1} \psi^0 \quad \text{with} \quad A := [\text{diag}(2\sigma - 1) - ss'].
\]

\(\text{'}\) denotes transposition. As noted by Hertel et al. (1991), the system is singular for \(n = 2\). There is either no solution in this case or infinitely many, if \(\psi\) happens to be proportional to \(s\). Hence, let us assume \(n > 2\). The system can be shown to be non-singular for \(n > 2\).

Similarly, the solution of (9) is

\[
\alpha = M^{-1} b \quad \text{with} \quad M := [\text{diag}(1 - \sigma) + ea'],
\]

\(e_i := 1\), \(a_i := s^0_i \sigma_i\) and \(b_i := \eta^0_i - \sigma_i + \bar{\sigma}\). This system can be shown to be non-singular, if either \(\sigma_i = 1\) for just one \(i\) or \(\sigma_i \neq 1\) for all \(i\) and \(\sum_i s_i \sigma_i / (\sigma_i - 1) \neq 1\). In the singular case either no solution exists or infinitely many exist. The latter is the case if \(b\) happens to be in the subspace spanned by the columns of \(M\). Singularity is a “non-generic” event in the sense that it has measure zero in parameter space. One should be aware, however, that even though singularity would almost never be encountered in practice, one could face \(\alpha\)-parameters escaping to extreme values if either more than one \(\sigma_i\) approaches one or if \(\sum_i s_i \sigma_i / (\sigma_i - 1)\) approaches one. It is also worth noting that the restriction \(\sigma \leq 1\) may needlessly generate extra problems with solving (9), because the upper bound \(\sigma_i = 1\) may well be attained for more than one \(i\), making (9) insoluble.
4 Flexibility of the CDES system

As mentioned, the CDES system is not fully flexible. Is it at last semiflexible in the sense that own-price elasticities and income elasticities can be reproduced as long as they are regular? Observed income and own-price elasticities are called regular if they can be conceived as stemming from a regular demand system. This is the case if and only if Engel aggregation holds, that is \( \sum_i s_i \eta_i = 1 \) and own-price elasticities are nonpositive.

Unfortunately, CDES is not even semiflexible, even after removing the unnecessary parameter restriction, because regular elasticities can produce negative \( \sigma \)- and \( \alpha \)-parameters solving equations (7) and (9). While nonnegative \( \sigma \)-parameters always generate nonpositive own-price elasticities according to equation (7),\(^4\) the inverse statement does not hold. As a case in point, take the example in Huff et al. (1997, p. 140) for an aggregate three sectors economy. Tables 1 to 3 show the expenditure shares and compensated own price elasticities in the first two columns under \( s^0 \) and \( \psi^0 \) for USA, EU and Rest of World, respectively. The \( \sigma^0 \)-column shows the solutions of (7), leading to irregularity in all three cases, because at least one \( \sigma_i \) is in all three cases negative. Furthermore, one \( \sigma_i \)-parameter exceeds one in all cases. For the USA, none of the \( \sigma \)-parameters lies between zero and one, the range imposed by Huff et al. (1997).

One has to adjust \( \sigma \) for attaining regularity, even if the unnecessary restriction on the \( \sigma_i \)s to be all on the same side of one is not imposed. The natural way out of the calamity is to minimize a distance between \( \psi \) and the expression on the right hand side of (7), subject to the constraint \( \sigma \geq 0 \).

Let the distance be \( \sum_i d(-\psi_{ii} : -\psi_{ii}^0) \), with \( d(q : r) \) one of the following three (assum-

\(^4\)Hertel et al. (1991, p. 273) claim that a sufficiently large \( \sigma_i \) could generate a positive \( \psi_{ii} \), if \( s_i < 1/2 \). This is not correct. Regularity, proved above for \( \sigma \geq 0 \), implies non-positive diagonal elements of the AUES, and hence also of the own-price elasticity matrix. This can also be seen directly:

\[
\psi_{ii} = (-1 + 2s_i - s_i^2)\sigma_i - s_i \sum_{j \neq i} s_j \sigma_j
\]

\[
= -(1 - s_i)^2\sigma_i - s_i \sum_{j \neq i} s_j \sigma_j \leq 0.
\]
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Table 2: Calibration of $\sigma$, minimal WSSQ

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Table 3: Calibration of $\sigma$, minimal SSQ
\[ \text{Inf}(q : r) := \begin{cases} 
q \log(q/r) - q + r & \text{if } q > 0 \\
r & \text{if } q = 0 \\
\infty & \text{else}, 
\end{cases} \]

2. weighted sum of squares:

\[ \text{SSQW}(q : r) := (q - r)^2 / r, \]

3. (unweighted) sum of squares:

\[ \text{SSQ}(q : r) := (q - r)^2. \]

\( d \) is strictly convex in \( q \), nonnegative, and \( d(q : r) = 0 \) if and only if \( q = r \), for all three measures. Using Inf is equivalent to the approach suggested by Huff et al. (1997). Due to linearity of \( \psi \) in \( \sigma \) this is a convex problem with lower and possibly upper bounds (if the unnecessary restrictions are imposed) on the variables. It is therefore easy to solve. It is even simpler, namely quadratic, for SSQW and SSQ. The column under \( \sigma^* \) in Tables 1 to 3 show the outcome if we constrain \( \sigma \) only to be nonnegative. The columns under \( \psi^* \) show the corresponding compensated own-price elasticities. Columns under \( \sigma^\dagger \) and \( \psi^\dagger \) show the outcomes, if \( \sigma \) is constrained to lie between zero and one.

Several observations can be made. First, results are not sensitive with respect to the distance measure. This comforting statement has to be qualified, however. As to the measures Inf and SSQW the similarity doesn’t come as a surprise, because those measures are second-order identical at \( q = r \) and thus behave similarly. Whether these measures’ behaviour differs from that of SSQ depends on the range of the \( \psi_{ii} \)'s. If they are not very different, and if none of them comes close to zero, as it is in the example, the choice of measure does not matter much. Otherwise it could.

Second, it turns out that the best fitting \( \sigma \)-parameters never would all stay within the zero-one interval, if they were not forced to. Hence, the unnecessary restriction to keep the \( \sigma \)-parameters below one is always binding, when imposed.

Third, one might have the impression that the \( \sigma \)-restriction, though binding, is not too restrictive, because the own-price elasticities do not change dramatically, if the constraint is removed (compare \( \psi^* \) with \( \psi^\dagger \)). The \( \sigma \)-parameters, however, do change, and consequently the off-diagonal cross-price elasticities change as well. Table 4 reveals that off-diagonal elements of the AUES matrix change considerably, if the constraint \( \sigma \leq 1 \) is imposed, in addition to the \( \sigma \geq 0 \) constraint. In particular, the AUES between food and manufactures is dramatically reduced by imposing the constraint.

In the discussed example the number of commodities is small and one of them has a share even larger than 1/2. The first term on the right hand side of (7) has therefore a large impact. In practice the number of commodities is often much larger, and their respective shares in the budget tend to be small. Equation (7) is then dominated by the second term. \( \psi_{ii} \) and \( \sigma_i \) tend to be equal, if \( s_i \) gets small. In practice formula (7)
Table 4: AUES with with and without upper bound on $\sigma$

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will therefore most likely deliver positive $\sigma$-parameters, and no modification is required. However, own price elasticities may well exceed one in absolute value, and the implied $\sigma$-parameter obtained by formula (7) will do so as well. In this situation, imposing the $\sigma \leq 1$ restriction forces the own-price elasticity to differ from the observations, even though the observations could be reproduced one-to-one by a globally regular CDES demand system!

Unfortunately, (9) is not guaranteed to deliver positive $\alpha$-parameters either. At any rate, $\alpha$ will be positive if the $\eta_i$s do not differ too much, which by Engel-aggregation means they do not differ too much from one. To see this, observe that for $\alpha = 1$ we have $\alpha_i \sigma_i = \sigma_i$ for all $i$ and hence $\alpha \sigma = \bar{\sigma}$ and $\eta_i = \alpha_i = 1$ for all $i$. Thus we may write (10) as $\alpha - 1 = M^{-1}(\eta^0 - 1)$; deviations of $\alpha_i$s from one are linear in deviations of $\eta_i$s from one; the former vanish if the latter vanish. General insights into what makes the range of $\eta_i$s shrink or amplify under the transformation are however hard to obtain.

If $\alpha$-parameters solving (9) become negative, a similar procedure as explained for calibrating $\sigma$-parameters is required. This is illustrated in Tables 5 to 7. We minimize the SSQ-distance between $\eta^0$ and $\eta = M(\alpha - 1) + 1$, subject to $\alpha \geq 0$ and $s^0 \alpha = 1$. The other distance measures deliver virtually identical results. Tables 5, 6 and 7 show results obtained with $\sigma$-calibrations from Tables 1, 2 and 3, respectively. In the example we assume an income-elasticity for service equal to 1.1 and equal elasticities for the other two sectors, shown under $\eta^0$. $\alpha^0*$ and $\alpha^0$ denote $\alpha$-Parameters obtained with $\sigma$-Parameters without and with upper bound, respectively. The corresponding modified nonnegative $\alpha$-Parameters are $\alpha^*$ and $\alpha^\dagger$. $\eta^*$ and $\eta^\dagger$ are the corresponding elasticities.

We observe that the implied nonmodified elasticities are highly sensitive with respect to the data. In one case, namely for the USA with $\sigma$-parameters not bounded from above, the implied $\alpha$-Parameters are extreme, because the respective $M$-matrix is coming close to singularity. Though this instability does not carry over to the modified parameters, the whole procedure turns out to be rather problematic. We find for example dramatically different parameters for the USA and the ROW, even though the basic data on shares and elasticities are not that different. What is also striking is that the $\eta$-calibration strongly depends on whether we apply the upper bound on $\sigma$ in the $\sigma$-calibration. For the USA, not only the values, the entire pattern of parameters is completely different between the two cases. It is therefore important whether we restrict the $\sigma$-parameters to the zero-one
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Table 5: Calibration of $\eta$ with $\sigma$ from Table 1

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Table 6: Calibration of $\eta$ with $\sigma$ from Table 2

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Table 7: Calibration of $\eta$ with $\sigma$ from Table 3
range or only to nonnegativity, as recommended in this article.

5 Equivalence of calibrated and Hanoch form

It remains to be shown how our calibrated form of the CDES system as specified in equation (2) is related to Hanoch’s original representation. In this respect the answer depends on whether we refer to Hanoch’s 1971 or 1975 papers in *Econometrica*. The CDES is dealt with explicitly only in the latter. The former is confined to the CRESH. Once we have specified the CRESH, however, the definition of CDES is obvious; we just have to substitute $x_i^{\alpha_i}y_i$ for the quantities $x_i$. Hence, we can refer to the 1971 paper, even though it does not treat the CDES explicitly.

The fundamental difference between the cited papers is that the restriction requiring the $\delta_i$s to have a uniform sign only appears in the 1975 paper, not in the 1971 paper. The latter is the one we refer to now. It is almost equivalent to our specification. What means “almost” in this context will be explained.

The parameter restrictions in the 1971 paper are (see Hanoch (1971, p. 697 and p. 700, footnote 13)):

1. $\delta_i \leq 1$ for all $i$;
2. either $B_i\delta_i > 0$ or $B_i\delta_i < 0$ for all $i$;
3. $B_i > 0$ for at least one $i$;
4. $\alpha_i \geq 0$ for all $i$ and $\alpha_i > 0$ for at least on $i$.

The conditions imply $\delta_i \neq 0$ for all $i$. The case $\delta_i = 0$ is also dealt with, but we omit it for the sake of brevity. Restriction 4 is added here, because the 1971 paper only deals with the homothetic case. In the 1975 paper and all later applications I am aware of, $B_i > 0$ is required for all $i$, implying that either $\delta < 0$ for all $i$ or $\delta > 0$ for all $i$. By the Hanoch-form we now mean the form (1) with parameter restrictions 1 to 4, while the calibrated form is (2) with parameter restrictions stated on page 5.

We first prove that to any Hanoch-form there is an equivalent calibrated form in the sense that both define the same indirect utility. For a given Hanoch-form, choose an arbitrary $y^0 \gg 0$ with

$$\sum_i B_i(y^0_i)^{\delta_i} = 1. \quad (11)$$

It always exists, as can be seen as follows. Let $I_+$ and $I_-$ denote the sets of $i$ with $B_i > 0$ and $B_i < 0$, respectively. $I_+$ is non-empty by assumption, while $I_-$ can be empty. (11) is rewritten as

$$\sum_{i \in I_+} B_i(y^0_i)^{\delta_i} = 1 - \sum_{i \in I_-} B_i(y^0_i)^{\delta_i}. $$

By an appropriate choice of $y^0$, the left hand side can attain any positive number, while the right hand side equals one if $I_-$ is empty, and can attain any number larger than one otherwise. Hence, both sides can be made equal by an appropriate choice of $y^0$.

Furthermore, choose

$$s^0_i = \frac{\delta_i B_i(y^0_i)^{\delta_i}}{\sum_j \delta_j B_j(y^0_j)^{\delta_j}}. $$
As $\delta_i B_i < 0$ for all $i$ or $\delta_i B_i > 0$ for all $i$, $s_i^0 > 0$; $\sum_i s_i^0 = 1$ is obvious. Now inserting the Box-Cox definition into (2) gives

$$\sum_i \frac{s_i^0}{\delta_i} \left[ u^{\alpha_i \delta_i} \left( y_i^0 / y_i^0 \right)^{\delta_i} - 1 \right] = 0. \quad (12)$$

Inserting $s_i^0$ and multiplying by $\sum_j \delta_j B_j (y_j^0)^{\delta_j} \neq 0$ shows this to be equivalent to

$$\sum_i B_i u^{\alpha_i \delta_i} y_i^{\delta_i} = \sum_i B_i (y_i^0)^{\delta_i} = 1.$$

As to the inverse statement: given a calibrated form, is there always an equivalent Hanoch-form? Yes, except if $D := \sum_j s_j / \delta_j$ happens to vanish. Choose

$$B_i = \frac{s_i^0 (y_i^0)^{-\delta_i}}{D \delta_i}.$$

$B_i$ exists, and if $D \neq 0$, either $B_i \delta_i > 0$ for all $i$ (if $D > 0$) or $B_i \delta_i < 0$ for all $i$ (if $D < 0$). $B_i > 0$ for at least one $i$, because either all $\delta_i$ have the same sign implying $\delta_i D > 0$ and hence $B_i > 0$ for all $i$, or they have different signs, implying that at least one $B_i$ is positive.

Substituting $DB_i (y_i^0)^{\delta_i}$ for $s_i^0 / \delta_i$ in (12) shows that (12) is equivalent to

$$\sum_i DB_i u^{\alpha_i \delta_i} y_i^{\delta_i} = D$$

which for $D \neq 0$ is equivalent to the Hanoch-form.

There is thus the possibility that for a given calibrated form an equivalent Hanoch-form with finite $B$-parameters cannot be found; namely the case $D = 0$. But this case has measure zero in parameter space, it is non-generic. In this sense, there is an equivalent calibrated form to any Hanoch-form, and an equivalent Hanoch-form to almost any calibrated form. This means that the calibrated form is slightly more general than the Hanoch-form. Though in practice we would almost never want parameters to fulfill $D = 0$ exactly, it may well be the case that $D$ comes close to zero. This implies large absolute values of the corresponding $B$-parameters. The calibrated form is in this situation probably numerically more stable. This is a further advantage of the calibrated form.

6 Conclusion

We conclude that CDES is a globally regular demand system not requiring the so-called substitution parameters all to have the same sign. The parameter restrictions originally imposed on the CRESH function in Hanoch (1971) are sufficient for global regularity. They require substitution parameters to be less or equal to one, expansion parameters to be nonnegative and not all zero, and distribution parameters to have the following sign pattern: At least one of them must be positive, and they must all have either the same or all have the opposite sign as the corresponding substitution parameter.

We have also shown that with the help of the Box-Cox transform the CDES can be presented in a more comfortable form. The substitution and expansion parameters are the same as in the original Hanoch-form. Instead of the distribution parameters with the awkward sign restrictions just described, benchmark expenditure shares and normalised prices appear as parameters. They can be read directly in a benchmark data set.
Appendix

We derive (4) by derivating $\log s_i$ according to (3) with respect to $\log p_j$, taking into account that $m$ is a function of $p_j$. Denoting $D$ the denominator in (3) and observing that

$$\frac{\partial \log D}{\partial \log p_j} = s_j \delta_j$$

and

$$\frac{\partial \log D}{\partial \log m} = -\sum_j s_j \delta_j =: -\bar{\delta}$$

we find

$$\frac{d \log s_i}{d \log p_j} = \frac{\partial \log s_i}{\partial \log p_j} + \frac{\partial \log s_i}{\partial \log m} \frac{\partial \log m}{\partial \log p_j}$$

$$= (\Delta_{ij} \delta_i - s_j \delta_j) + (\bar{\delta} - \delta_i) s_j.$$ 

For $h_i = s_i m/p_i$ we then have

$$\frac{d \log h_i}{d \log p_j} = \frac{d \log s_i}{d \log p_j} + s_j - \Delta_{ij}.$$ 

Putting the two equations together yields (4).

For deriving (6) we derivate $\log s_i$ according to (3) with respect to $\log m$, taking into account that $u$ is a function of $m$, implicitly given by (2). Observing that

$$\frac{\partial \log D}{\partial \log u} = \sum_i s_i \delta_i \alpha_i$$

$$= \sum_i s_i \alpha_i (1 - \sigma_i)$$

$$= \bar{\alpha} - \alpha \sigma,$$

and using (13) yields

$$\frac{d \log s_i}{d \log m} = \frac{\partial \log s_i}{\partial \log m} + \frac{\partial \log s_i}{\partial \log u} \frac{d \log m}{d \log u}$$

$$= (\bar{\delta} - \delta_i) + (\alpha_i \delta_i - \bar{\alpha} + \alpha \sigma) \frac{d \log u}{d \log m}.$$ 

To find the last term, the elasticity of utility with respect to expenditure is implicitly given by (2); insert $z_i := u^{\alpha_i}$ into (2). It is then obvious that

$$\frac{\partial \log m}{\partial \log z_i} = \frac{\partial \log m}{\partial \log p_i} = s_i,$$

and hence

$$\frac{d \log m}{d \log u} = \sum_i \frac{\partial \log m}{\partial \log p_i} \frac{d \log z_i}{d \log u}$$

$$= \sum_i s_i \alpha_i$$

$$= \bar{\alpha}.$$
This gives us

\[
\eta_i = \frac{d \log s_i}{d \log m} + 1
= (\tilde{\delta} - \delta_i) + (\alpha_i \delta_i - \bar{\alpha} + \sigma)/\bar{\alpha} + 1
= (\sigma_i - \bar{\sigma}) + (\sigma_i - \sigma) + (\alpha_i + \sigma - \alpha_i \sigma)/\bar{\alpha}.
\]

**References**


