Notes on Final Demand in the Presence of Non-homothetic, Weak Separability
T. Hertel, February 2001

Introduction

These notes are inspired by McDougall's (2001) Technical Paper in which he develops a new final demand system for GTAP. They are intended to provide an overview of some of his main findings as well as a discussion of empirical implications.

The Utility Elasticity of Expenditure

The fundamental problem being addressed by McDougall is one in which household demands (in this case associated with the regional household) may be written as a function of a set of G composite sub-utility aggregates: $U(U_1,\ldots,U_G)$, but where one or more of these aggregates may exhibit non-homotheticity. Therefore, we cannot define a unique price index for the non-homothetic utility sub-aggregate, since the "price" of attaining another unit of utility will depend on the level of utility. This forces us to introduce into the model a new variable, namely the elasticity of expenditure with respect to utility, which McDougall labels Φ.

McDougall introduces the Lagrangean associated with this constrained optimization problem:

$$\mathfrak{L} = U(U_1,\ldots,U_G) - \Lambda\left(\sum_i E_i(P_i,U_i) - X\right) \tag{1}$$

here X is regional household income or expenditure, and $E_i(P_i,U_i)$ is the expenditure function associated with utility aggregate i – e.g., private household demands. Logarithmic differentiation of (1) with respect to X, and application of the envelope theorem, gives the elasticity of utility with respect to expenditure, which is the inverse of the elasticity of expenditure with respect to utility.

$$\begin{aligned} \partial \log U/\partial \log X &= (X/U)\,\partial\mathfrak{L}/\partial X \\ &= (X/U)\Lambda \\ &= \Phi^{-1} \end{aligned} \tag{2}$$

Similarly, logarithmic differentiation of (1) with respect to the price of good j entering into the ith composite category and application of the envelope theorem, followed by Shepherd's lemma gives:

$$\partial \log U/\partial \log P_{ij} = \frac{P_{ij}}{U}\frac{\partial \mathfrak{L}}{\partial P_{ij}} = -\Phi^{-1} S_{i_j} \tag{3}$$

where S_{ij} is the share in total regional expenditure of commodity j used in composite i (e.g., food used in private consumption).

We are now in a position to totally differentiate the regional household's indirect utility function $U(P,X)$:

$$u = \sum_i \sum_j (\partial \log U/\partial \log P_{ij})\,p_{ij} + (\partial \log U/\partial \log X)\,x \tag{4}$$

where lower case denotes percentage change. Using (2) and (3) we have:

$$u = -\Phi^{-1}\sum_i \sum_j S_{ij}\, p_{ij} + \Phi^{-1} x \qquad (5)$$
$$u = \Phi^{-1}(x - p)$$

where p is the expenditure-share weighted index of individual commodity price changes.

Therefore:

$$x = p + \Phi\, u \qquad (6)$$

Equation (6) is the fundamental equation for determining utility in McDougall's formulation. It is clear that the relationship between utility and expenditure is no longer constant and this will have implications for both demand and welfare determination.

The Cobb-Douglas Case

So what does the utility elasticity of cost look like? This obviously depends on the functional form chosen for regional household utility. In the GTAP model, this is the Cobb-Douglas form:

$$U = C \prod_i U_i^{B_i} \qquad (7)$$

In this case, the elasticity of utility with respect to expenditure is given by:

$$\Phi^{-1} = \sum_j \Phi_j^{-1} B_j \qquad (8)$$

where Φ_j^{-1} is the elasticity of utility with respect to expenditure in the j[th] aggregator function (e.g., private household demand). McDougall shows that the resulting expenditure share equations are given by:

$$(X_i/X) = \Phi_i^{-1} B_i / \Phi^{-1} \qquad (9)$$

Clearly when the utility elasticity of cost is non-constant, the expenditure shares in the demand system given by (9) will not be constant. This is a marked departure from the usual Cobb-Douglas case obtained when composite prices are exogenous.

A Focus on Private Demands and the CDE Function

In the GTAP model, the composite goods consumed by the regional household are savings, government purchases, and private purchases. Savings are represented by a single commodity, so $\Phi_{savings} = 1$. Government demands are modeled using the Cobb Douglas form, so they too are homothetic and $\Phi_{Govt} = 1$. The problem (and hence the need for these notes!) derives from private consumption where the non-homothetic, CDE function is used. Here, McDougall refers to the paper by the inventor of the CDE functional form, Giora Hanoch, which shows that:

$$\Phi_p = \sum_i S_{pi}\,\gamma_i \tag{10}$$

where S_{pi} represents the private expenditure share on good i and γ_i represents the good i expansion parameter in the CDE functional form. The value of Φ_p is set equal to one when the CDE is initially calibrated to the GTAP data. However, when the γ_i*'s* differ by commodity, then changes in the private consumption shares, S_{pi}, will interact with the γ_i*'s* to lead to changes in Φ_p. In particular, when the consumption share of a good with a low γ_i (a necessity) rises, then Φ_p will fall.

In order to see more clearly how this works, totally differentiate (10) to get:

$$\varphi_p = \sum_i S_{\gamma i}(x_{pi} - x_p) = \sum_i S_{\gamma i}\,(p_{pi} + q_{pi} - x_p) \tag{11}$$

where φ_p is the percentage change in Φ_p, x_{pi}, and x_p are the percentage changes in expenditures on good i and total private expenditure, respectively. The coefficient $S_{\gamma i} = S_{pi}\,\gamma_i / \sum_j \gamma_j S_{pj}$.

We make one further modification to (11) in order to facilitate interpretation. Because the sum of the private household consumption shares must always equal one, the total differential of these shares must equal zero (i.e., if one share rises, another must fall, by definition of the share relationship). Therefore the share weighted changes must equal zero, so that:

$$\sum_i S_{pi} s_{pi} = \sum_i S_{pi}(x_{pi} - x_p) = \sum_i S_{pi}(p_{pi} + q_{pi} - x_p) = 0 \tag{12}$$

Making use of this fact, we can factor S_{pi} out of the $S_{\gamma i}$ term in (11), then add zero (i.e., $\sum_i S_{pi} s_{pi}$) to both sides of (11) to get:

$$\varphi_p = \sum_i S_{pi}\,[(\gamma_i / \sum_j \gamma_j S_{pj}) - 1]\,[p_{pi} + q_{pi} - x_p] \tag{13}$$

From (13) it can be seen that if the commodity whose share rises (i.e. for which the second term in brackets [·] > 0), also has an above-average expansion parameter (first term in brackets [·] > 0), then $\varphi_p > 0$. On the other hand, if the private expenditure share for necessities (below-average γ_i) rises, then $\varphi_p < 0$.

Implications for Regional Household Demands

What does all this imply for the behavior of aggregate demands? Returning to (9), and making use of the fact that $\Phi_G = \Phi_S = 1$, we have the following share equations for the regional household demands:

$$
\begin{aligned}
(X_p/X) &= \Phi_p^{-1} B_p/\Phi^{-1} \\
(X_G/X) &= B_G/\Phi^{-1} \\
(X_S/X) &= B_s/\Phi^{-1} \\
\Phi^{-1} &= \sum_j \Phi_j^{-1} B_j = \Phi_p^{-1} B_p + B_G + B_s
\end{aligned}
\qquad (14)
$$

In terms of percentage changes this system becomes:

$$
\begin{aligned}
(x_p - x) &= -(\varphi_p - \varphi) = -(1 - S_p)\,\varphi_p \\
(x_G - x) &= S_p \varphi_p \\
(x_S - x) &= S_p \varphi_p \\
\varphi_p &= \sum_i S_{pi}\,[(\gamma_i / \sum_j \gamma_j s_{p_j}) - 1]\,[p_{pi} + q_{pi} - x_p]
\end{aligned}
\qquad (15)
$$

Empirical Illustration

Now we are in an excellent position to evaluate the impact on the regional household of a perturbation to the economy. Consider two different types: an income shock and a price shock. For the sake of convenience, let us use a partial equilibrium closure wherein supply is perfectly elastic (fixed prices) and income is exogenous. The data base is the base version found in OneGTAP.

Let us first consider the impact of doubling regional income. With the rise in *per capita* expenditure, the demands shares for government purchases are unaffected, but private demand shares are significantly altered. Taking the 3 good, one region example from OneGTAP, we have γ_i values of 0.316 for food, 0.840 for manufactures, and 1.215 for services. This translates into income elasticities of 0.5, 1.0, and 1.1, respectively. So, when *per capita* income doubles, the share spent on food falls, and that spent on services rises. (From Table 1 note that food demand rises by only 37%, whereas services demand rises by 112.5%.) So the shares changes interact positively with the expansion parameters γ_i and $\varphi_p > 0$ by equation (13). ($\varphi_p = 4.4\%$ in the first column of Table 1). This feeds directly into the regional household's demand system (15), causing $(x_p - x)$ to fall and $(x_G - x), (x_S - x)$ to rise. (Note that $x = 100\%$ in this case, whereas $x_p = 97.5\%$ in Table 1.

Thus, as the regional household becomes wealthier, this feature of the non-homothetic, separable demand system causes a shift towards government consumption and savings $(x_G = x_S = 106.2\%)$. Similar logic dictates that if a region were to suddenly become poorer, it would shift expenditures away from public goods and services (government) and future consumption (savings), towards current period, private consumption.

Table 1 Results from Two Partial Equilibrium Simulations[1]

Variable	Experiment[2]	
(% Change)	Doubling Income	10% Food Price Shock
Private Demands: q_{pi}		
food	37.0	-2.2
mnfcs	96.3	-1.8
svces	112.5	-0.9
Utility elasticity of private expenditure φ_p	4.4	-0.9
Regional expenditure x_i		
private consumption	97.5	0.3
government purchases	106.2	-0.6
savings	106.2	-0.6
Utility elasticity of regional expenditure φ	3.1	-0.6

1. Land version of One-GTAP used. Closure involves fixing market prices and regional income.

2. Solution method used = Gragg 2 4 6, automatic accuracy.

The second column in Table 1 reports the results of a 10% food price shock. In this case, food demand falls by 2.2%, which is not enough to offset the impact of a 10% price hike on expenditure. Meanwhile, with manufactures and services prices constant, declines in those quantities dictate a decline in their expenditure shares. This causes the utility elasticity of private expenditure to fall (0.9%), leading to a shift of regional expenditure towards private consumption. (The food price rise has made the regional household poorer.)

Summary and Implications for AGE Analysis

In summary, McDougall's new treatment of non-homothetic separability fixes a problem that has existed hitherto in the GTAP model. True optimization by the regional household must take into account the non-constant elasticity of expenditure with respect to utility. In particular, as the regional household becomes wealthier, there is a gradual shift in expenditure shares away from private consumption and towards public consumption (government) and future consumption (savings) for most policy simulations, in which income changes are quite modest, the resulting change in this elasticity – and hence in regional expenditure shares – are also quite small.

References

Hanoch, G. 1975. "Production and Demand Models with Direct or indirect, Implicit Additivity," Econometrica (43): 395-419

McDougall, R. 2001. "A New Regional Household Demand System for GTAP," GTAP Technical Paper No. 20.