A New Regional Household Demand System for GTAP

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Abstract

The GTAP final demand system has some known defects: the computation of the equivalent variation is not exact; with non-standard demand parameters, the equivalent variation may be grossly in error; the decomposition of the equivalent variation contains a nuisance term. We find a further defect, that the upper-level demand equations are invalid. We revise the model to remove these defects.

Contents

1 Introduction

The GTAP model, versions 4.1 and lower, suffers from some defects in the implementation of the regional household demand system:

- The upper level of the demand system assumes a fixed price for utility from private consumption. But with a private consumption demand system of the CDE form, the price of utility from private consumption depends on the level of private consumption expenditure. With households not price takers for utility from private consumption, the familiar Cobb-Douglas demand system does not apply. Accordingly, the upper-level demand equations are in error.
- Utility and equivalent variation are wrongly computed in simulations with non-standard settings for the CDE expansion parameters. Even with the standard settings, in multi-step simulations the utility and equivalent variation computations are inexact.
- The welfare decomposition inherits the defects of the equivalent variation computation.

In removing these defects we revise in passing some minor misfeatures of the old treatment:

- We treat the entire final demand system as the demand system of a representative household, rather as than a conglomeration of representative and region-wide demand systems (subsection 2.8).
- We provide a new facility for shifting the allocation of regional income exogenously by modifying rather than overriding the final demand system (subsection 2.15).
- We eliminate an uninterpretable "nuisance term" from the decomposition of equivalent variation (subsection 4.3).
- We reorganize the GTAP model source code to gather within a single module all equations derived from the upper-level demand system (subsection 2.8).

The main disadvantage of the new treatment relative to the old is that its implementation and properties are somewhat more complex. It requires more equations and variables, mostly to support an exact calculation of the equivalent variation. Also, whereas the old treatment allocated regional income in fixed shares between private consumption expenditure, government household expenditure, and saving, the new treatment allows the shares to vary in response to changes in income and consumer prices.

This paper describes the new treatment. It should be accompanied by several program files—source code, `gtap.tab`, for the revised solution model, a Tablo stored input file `gtap.fts` showing a typical model condensation, and a GTAP test simulation command file `ghom.cmf` showing a typical model closure. It includes extensive listings from the source code. It does not however describe but rather takes as given the standard GTAP model notation; it should accordingly be read in conjunction with the source code or the original GTAP model documentation (Hertel and Tsigas [5]).

Listings in this paper of old program code come from a GTAP model version 5 prerelease, incorporating Ken Itakura's reorganization of the code structure, but no relevant changes in the model theory over version 4.1, and used in the August 2000 GTAP short course.

We adopt the convention that a lower-case symbol denotes percentage change in the corresponding upper-case symbol; so, for a variable X, x denotes percentage change in X, $x = (1/100)(d\,X/X)$.

2 The upper level of the regional household demand system

2.1 The old treatment

In the GTAP model as originally implemented (Hertel and Tsigas [5]), in each region a *regional household* allocates regional income so as to maximize *per capita* aggregate utility according to a Cobb-Douglas utility function. The maximand is described as "aggregate" utility because it comprises both government and private sector behavior. The arguments in the utility function are *per capita* utility from private consumption, *per capita* utility from government consumption, and *per capita* real saving. We refer to these as the *upper-level commodities* of the final demand system.

Real saving is a single commodity, defined as saving deflated by a saving price. Utility from government consumption is a Cobb-Douglas aggregate of government consumption of individual commodities. *Per capita* utility from private consumption is aggregated from *per capita* private consumption of individual commodities following Hanoch's ([3]) constant difference elasticity (*CDE*) demand system.

We note that in the private consumption demand system, unlike the government consumption demand system, the variable maximized is a *per* *capita* rather than an economy-wide utility. This is necessary because the private consumption demand system is non-homothetic. The allocation of private consumption expenditure across commodities depends on the sum to be allocated, and the appropriate sum variable is not economy-wide but *per capita* private consumption expenditure.

The CDE demand system is characterized by an implicit expenditure-*cum*-indirect-utility function,

$$1 = \sum_i B_i U^{\Upsilon_i E_i} \left(\frac{P_i}{X}\right)^{\Upsilon_i}, \tag{2.1.1}$$

where U denotes utility, P_i, the price of commodity i, X, expenditure, and B_i, Υ_i, and E_i, various parameters. Following Hanoch [3], we call the B_i distribution parameters, the Υ_i substitution parameters, and the E_i expansion parameters. Constraints on the parameters are:

$$\forall i,\ B_i > 0,$$
$$\forall i,\ E_i > 0,$$

and either

$$\forall i,\ \Upsilon_i < 0$$

or

$$\forall i,\ 0 < \Upsilon_i < 1.$$

Although we are not required to do so by theory, in standard GTAP data bases we normalize the expansion parameters so that their share-weighted sum is equal to one,

$$\sum_i S_i^P E_i = 1,$$

where S_i^P denotes the share of commodity i in private consumption expenditure.

This completes the specification the final demand system; it remains to work out the implications of the specification. This is done briefly in Hertel and Tsigas ([5]), but to support later discussion (subsection 2.2) we provide here a more detailed derivation for the upper level of the system.

We write the upper-level utility function as

$$U = C U_P^{B_P} U_G^{B_G} U_S^{B_S}, \tag{2.1.2}$$

where U denotes *per capita* aggregate utility, U_P, *per capita* utility from private consumption, U_G, *per capita* government consumption, and U_S, *per capita* real saving, and B_P, B_G, and B_S are distribution parameters.

We define a saving price P_S, and postulate the existence of suitable price indices P_G and P_P for utility from government and private consumption. Then given income Y, the regional household maximizes U subject to the budget constraint

$$N(P_P U_P + P_G U_G + P_S U_S) = Y, \tag{2.1.3}$$

where N denotes population.

Since the utility function is Cobb-Douglas, we expect the regional household to allocate regional income in fixed shares between the upper-level commodities:

$$Y_P = \frac{B_P}{B} Y, \tag{2.1.4}$$

$$Y_G = \frac{B_G}{B} Y, \tag{2.1.5}$$

$$Y_S = \frac{B_S}{B} Y, \tag{2.1.6}$$

where B denotes the sum of the distribution parameters, $B = B_P + B_G + B_S$, Y_P private consumption expenditure, $Y_P = NP_P U_P$, Y_G government consumption expenditure, $Y_G = NP_G U_G$, and Y_S saving, $Y_S = NP_S U_S$. Then

$$NP_P U_P = \frac{B_P}{B} Y,$$
$$NP_G U_G = \frac{B_G}{B} Y,$$
$$NP_S U_S = \frac{B_S}{B} Y.$$

Putting

$$Q_P = NU_P, \tag{2.1.7}$$

$$Q_G = NU_G, \tag{2.1.8}$$

$$Q_S = NU_S, \tag{2.1.9}$$

, where Q_P denotes private consumption, Q_G government consumption, and

Q_S saving, we obtain

$$P_P Q_P = \frac{B_P}{B} Y,$$
$$P_G Q_G = \frac{B_G}{B} Y,$$
$$P_S Q_S = \frac{B_S}{B} Y.$$

To allow for exogenous shocks in the allocation of saving, we define "slack variables" K_S and K_G for saving and utility from government consumption, initially equal to one. We insert these into the corresponding demand equations:

$$P_G Q_G = K_G \frac{B_G}{B} Y,$$
$$P_S Q_S = K_S \frac{B_S}{B} Y.$$

Differentiating and rearranging, we obtain

$$q_G = y - p_G + \kappa_G, \tag{2.1.10}$$
$$q_S = y - p_S + \kappa_S. \tag{2.1.11}$$

These appear in the old code as:

```
Equation GOVERTU
# computation of utility from government consumption (HT 39) #
(all,r,REG)
    ug(r) = y(r) - pgov(r) + govslack(r);
```

and

```
Equation SAVINGS
# regional demand for savings (HT 38) #
(all,r,REG)
    qsave(r) = y(r) - psave(r) + saveslack(r) ;
```

In the presence of shocks to the slack variables, the upper-level demand system is no longer operative; the budget constraint however must still be observed. Accordingly, we include in the model not the demand equation for utility from private consumption but instead the budget constraint

$$Y_P = Y - Y_G - Y_S.$$

For no compelling reason, we express government consumption expenditure Y_G as the sum of expenditures on individual commodities, $Y_G = \sum_i Y_{Gi}$, where Y_{Gi} denotes government consumption expenditure on commodity i, $Y_{Gi} = P_{Gi} Q_{Gi}$, where P_{Gi} denotes the price of commodity i when purchased for government consumption, and Q_{Gi} government consumption of commodity i. Then

$$\begin{aligned} Y_P &= Y - \sum_i Y_{Gi} - Y_S \\ &= Y - \sum_i P_{Gi} Q_{Gi} - P_S Q_S, \end{aligned}$$

or, in percentage change form,

$$Y_P y_P = Y y - \sum_i Y_{Gi}(p_{Gi} + q_{Gi}) - Y_S(p_S + q_S).$$

This appears in the old code as:

```
Equation PRIVATEXP
# private consumption expenditure (HT 8) # (all,r,REG)
    PRIVEXP(r)*yp(r)
  = INCOME(r)*y(r)
  - SAVE(r)*[psave(r) + qsave(r)]
  - sum(i,TRAD_COMM, VGA(i,r)*[pg(i,r) + qg(i,r)])
  ;
```

Finally, we compute utility. Substituting for U_G and U_S from equations (2.1.8) and (2.1.9) into equation (2.1.2), we have

$$U = C U_P^{B_P} \left(\frac{Q_G}{N}\right)^{B_G} \left(\frac{Q_S}{N}\right)^{B_S}.$$

Differentiating, we obtain

$$\begin{aligned} u &= B_P u_P + B_G(q_G - n) + B_S(q_S - n) \qquad (2.1.12) \\ &= B\left[\frac{B_P}{B} u_P + \frac{B_G}{B}(q_G - n) + \frac{B_S}{B}(q_S - n)\right] \\ &= B\left[\frac{Y_P}{Y} u_P + \frac{Y_G}{Y}(q_G - n) + \frac{Y_S}{Y}(q_S - n)\right], \end{aligned}$$

using equations (2.1.4–2.1.6). Then, setting $B = 1$, we have

$$Y u = Y_P u_P + Y_G(q_G - n) + Y_S(q_S - n). \qquad (2.1.13)$$

This appears in the old code as:

```
Equation UTILITY
# computation of per capita regional utility (HT 37) #
(all,r,REG)
    INCOME(r)*u(r)
  = PRIVEXP(r)*up(r)
  + GOVEXP(r)*[ug(r) - pop(r)]
  + SAVE(r)*[qsave(r) - pop(r)]
  ;
```

2.2 Defects in the old treatment: initial findings

While the old treatment has proven serviceable in many GTAP applications, it has its defects. We identify three, of very different magnitude:

- It is slightly confusing in formulation, shifting unnecessarily between unitary and representative households, and *per capita* and economy-wide utilities.
- In setting saving or government consumption exogenously, the user cannot adjust preferences within the upper-level demand system, but must override them. There are some advantages to maintaining a working upper-level demand system even when some upper-level demands are exogenized.
- The underlying theory (subsection 2.1) is invalid; the model equations do not logically follow from the system specifications.

The first, and very minor, objection to the old treatment is that in formulation it is slightly incoherent. The upper-level utility function is attributed to a unitary "regional household", but its arguments are *per capita* variables. The government consumption variable in the upper-level utility function is *per capita* government consumption, but in the government consumption demand system the variable is economy-wide government consumption. Utility from private consumption pertains to a representative private household, and utility from government consumption to a "government household", both distinct from the "regional household" that enjoys aggregate utility. None of these disconnections is substantively damaging, but together they create a slight impediment to thinking and writing about the demand system.

If we take the descriptions in the old treatment seriously, we are not entitled to talk about upper and lower levels of the demand system. To do so implies that they are part of the same agent's demand system, whereas really they pertain to different agents. Strictly speaking, the aggregate utility function pertains to a unitary regional household that displays an altruistic interest in the welfare of the representative private household, and also cares about a variable, *per capita* government consumption, that is related to but distinct from the welfare of the government household. We deliberately slur over these niceities in deriving the old system (subsection 2.1). In discussing below (in this subsection) the more substantive defects of the old system, we override them, treating all the demand subsystems as components of a representative regional household demand system. Finally, in presenting the new treatment, we explicitly adopt the unified approach (subsection 2.8), and implement the associated minor substantive changes (subsection 2.10).

The second limitation of the old treatment, also minor, is that the saving and government consumption slack variables, K_S and K_G, override rather than modify the upper-level demand system. We should be able to represent exogenous shifts in income disposition as shifts in preferences in the upper-level demand system. This would have three advantages:

- It would let us shock demand for any of the three upper-level commodities. The old treatment lets us shock either saving or government consumption but not private consumption.

- It would let the upper-level demand system do some work even when some external outcomes are imposed. For example, while exogenizing saving, we could let the demand system allocate remaining income between private and government consumption.

- It would allow us to obtain meaningful welfare results even when some upper-level income allocations are set exogenously.

The main defect in the old treatment is that the theory it implements is invalid. The error is embodied in the old upper-level budget constraint (2.1.3), $N(P_P U_P + P_G U_G + P_S U_S) = Y$. In adopting this formulation for the constraint, we assume that the regional household can obtain utility from private consumption at some fixed price P_P. This assumption is non-trivial and in fact unwarranted.

We rewrite the old upper-level budget constraint as

$$P_P U_P + P_G U_G + P_S U_S = X, \qquad (2.2.1)$$

where X denotes *per capita* income. Recalling that utility from private consumption and utility from government consumption are defined within the private and government consumption demand subsystems, we obtain the general form of the constraint,

$$E_P(\mathbf{P_P}, U_P) + E_G(\mathbf{P_G}, U_G) + P_S U_S = X, \tag{2.2.2}$$

where E_P and E_G are *per capita* expenditure functions, and $\mathbf{P_P}$ and $\mathbf{P_G}$ price vectors, for private and government consumption. It might so happen that the expenditure functions were of the form

$$\begin{aligned} E_P(\mathbf{P_P}, U_P) &= \Pi_P(\mathbf{P_P})U_P, \\ E_G(\mathbf{P_G}, U_G) &= \Pi_G(\mathbf{P_G})U_G \end{aligned} \tag{2.2.3}$$

for some functions $\Pi_P(\mathbf{P_P})$ and $\Pi_G(\mathbf{P_G})$. If so, we could could set $P_P = \Pi_P(\mathbf{P_P})$ and $P_G = \Pi_G(\mathbf{P_G})$, and replace the general budget constraint (2.2.2) with the simpler form (2.2.1). In fact, the government consumption expenditure function is of the required form, but the private consumption expenditure function is not; so we cannot use the simpler budget constraint.

To show that the private consumption expenditure function cannot be written in the form (2.2.3), we employ the general proposition (cf., e.g. Deaton and Muellbauer [1] p. 143):

Proposition 1 *For any demand system, the expenditure function is of the form $E(\mathbf{P}, U) = \Pi(\mathbf{P})F(U)$ for some monotonic increasing function F if and only if the system is homothetic.*

Proof. For sufficiency, note that if the system is homothetic, there exists a strictly increasing function F such that for all consumption vectors $\mathbf{Q}$, for all positive K, $F \circ U(K\mathbf{Q}) = KF \circ U(\mathbf{Q})$. Let U_0 be some arbitrary utility level; and for all price vectors $\mathbf{P}$, let $\Pi(\mathbf{P}) = E(\mathbf{P}, U_0)/F(U_0)$. Then for any utility level U_1 and any price vector $\mathbf{P}$, $E(\mathbf{P}, U_1) = \Pi(\mathbf{P})F(U_1)$. For suppose that with prices $\mathbf{P}$, consumption $\mathbf{Q_0}$ yields utility U_0 at minimum cost. Then consumption $(F(U_1)/F(U_0))\mathbf{Q_0}$ yields utility U_1 at minimum cost; so $E(\mathbf{P}, U_1) = \mathbf{P} \bullet (F(U_1)/F(U_0))\mathbf{Q_0} = (F(U_1)/F(U_0))E(\mathbf{P}, U_0) = \Pi(\mathbf{P})F(U_1)$. Hence the expenditure function is of the specified form. For necessity, note that by Shephard's lemma, the budget share of each commodity i is equal to the elasticity of Π with respect to the price of i; so the budget shares are independent of utility; so the system is homothetic.■

Now as Hanoch [3] shows, the CDE is in general non-homothetic. Indeed, this is a requirement for any empirically satisfactory demand system (see, for example, Deaton and Muellbauer [1] p. 144), and part of the reason for adopting the CDE in GTAP (Hertel and Tsigas [5], p. 49). So in GTAP, the private consumption expenditure function is not of the form (2.2.3), and the budget constraint is not equation (2.2.1).

This shows that the old theory is defective, in that it contains an invalid derivation; it does not show how or whether the relevant results are in error. After dealing with a side-issue relating to the derivation (subsection 2.3) we correct the theory (subsection 2.4) and compare the corrected with the old results (subsection 2.5).

2.3 A digression on the Gorman conditions

As shown above (subsection 2.2), there is an error in the derivation of the old upper-level demand system. One way to view the error is that it imposes an erroneous two-stage budgeting scheme on the regional household demand system. It is natural then to inquire, how the defects of the old treatment relate to the necessary and sufficient conditions derived by Gorman [2] for the feasibility of two-stage budgeting.

The Gorman conditions apply in the context of *weak separability*. A system is said to be *weakly separable* if the utility function can be represented in the form

$$U(\mathbf{Q}) = U_\bullet(U_1(\mathbf{Q_1}), \ldots, U_G(\mathbf{Q_G})),$$

where $\mathbf{Q_1}, \ldots, \mathbf{Q_G}$ is a partition of the quantity vector $\mathbf{Q}$ into subvectors representing groups of commodities. The function $U_\bullet$ is the upper-level utility function, and the $U_1, \ldots, U_G$ are *lower-level utilities* or *subutilities*. In a weakly separable system, the Gorman conditions are the necessary and sufficient conditions for the existence of a upper-level demand system

$$\max U^*_\bullet(U^*_1, \ldots, U^*_G) \text{ subject to } \sum_{i=1}^{G} P^*_i U^*_i = X, \tag{2.3.1}$$

where P^*_i and U^*_i are price and quantity indices for the i'th lower-level subsystem, and $U^*_\bullet$ a *utility index*, such that the solution for the upper-level system is consistent with the solution for the overall system. Note that the quantity indices U^*_i may or may not be the subutilities U_i. Also, the utility index $U^*_\bullet$ may or may not be similar in form to the upper-level utility function $U_\bullet$.

By construction, the GTAP regional household demand system is weakly separable. The error in the old derivation is the assumption that utility from private consumption can serve as a quantity index for private consumption in the upper-level demand system (2.3.1). As shown above (subsection 2.2), if we try to use utility from private consumption as the quantity index, we find there is no corresponding price index. This does not mean that the regional household demand system does not meet the Gorman conditions; price and quantity indices for private consumption might yet be found; it means only that the quantity index cannot be the subutility.

On the other hand, even if suitable price and quantity indices did exist, that would not necessarily validate the old treatment. It would show that we could specify a upper-level demand system that treated private consumption as an ordinary good, but it would not guarantee that the utility index $U^*_\bullet$ in that system was of the Cobb-Douglas form, nor consequently that the demands had the Cobb-Douglas fixed budget shares property. In short, the requirements for the validity of the old treatment are more stringent than the Gorman conditions.

Gorman [2] shows that a upper-level system of the desired form can be constructed under either of two alternative conditions. One alternative is that the lower-level systems are homothetic. Under this alternative, the quantity indices are just the lower-level utilities, and the utility index is just the upper-level utility function. The other alternative is that the upper-level utility function is additive, $U_\bullet(U_1, \ldots, U_G) = \sum_{i=1}^{G} U_i$, and the lower-level systems admit indirect utility functions of the *Gorman generalized polar form*,

$$\Psi_i(\mathbf{P_i}, X_i) = F_i\left(\frac{X_i}{M_i(\mathbf{P_i})}\right) + A_i(\mathbf{P_i}). \tag{2.3.2}$$

Under this alternative, the quantity indices, $U_i^* = X_i/M_i(\mathbf{P_i})$, the price indices, $P_i^* = M_i(\mathbf{P_i})$, and the utility index, $U^*_\bullet(U_1^*, \ldots, U_G^*) = \sum_{i=1}^{G} F_i(U_i^*)$.

As we have seen already, the GTAP final regional household demand system does not meet the first condition, that the lower-level demand systems be homothetic. It seems obvious, but is not easily proved, that except in degenerate cases, the CDE and Gorman generalized polar forms are incompatible.

Conjecture 1 *If a demand system is both a CDE system and a Gorman generalized polar form, then it is a CES system.*

As noted above, it is possible to satisfy the Gorman conditions without validating the old treatment of the upper-level demand system. More specifically, solutions involving homothetic lower-level demand systems validate the old treatment, but solutions involving the Gorman generalized polar form do not. In particular, the old treatment specifies a utility function of the Cobb-Douglas form, but the solutions involving the Gorman generalized polar form require a utility function of the additive form.

On the one hand, it is true to say that the old treatment is erroneous because the CDE system does not satisfy the Gorman conditions. It is true because, if the old treatment were valid, the Gorman conditions would necessarily be satisfied. On the other hand, the Gorman conditions are a something of a distraction in this context. To show that the old derivation is invalid, we do not need to refer to the Gorman conditions; it is sufficient, and simpler, to show that the private consumption demand system is non-homothetic. Nevertheless, as we show below (subsection 2.7), although the Gorman result is not useful in refuting the old treatment, it is potentially useful in remedying its defects.

2.4 Revised theory

We find above (subsection 2.2) that we need to revise the budget constraint in the upper-level demand system from the special form (2.2.1), $P_P U_P + P_G U_G + P_S U_S = X$, to the more general form (2.2.2), $E_P(\mathbf{P_P}, U_P) + E_G(\mathbf{P_G}, U_G) + P_S U_S = X$. We now derive the demand equations, an equation for utility, and some auxiliary equations under this more general assumption.

As an aid to the reader, we distinguish these derived equations by enclosing them in $\boxed{\text{boxes}}$.

We begin by obtaining a general solution for the Cobb-Douglas demand system in the absence of fixed prices.

Proposition 2 *In the Cobb-Douglas demand system*

$$\max U = C \prod_i U_i^{B_i} \text{ subject to} \sum_i E_i(U_i) = X, \tag{2.4.1}$$

with expenditures $X_i = E_i(U_i)$ on individual commodities convex in quantities U_i, the budget shares

$$\frac{X_i}{X} = \frac{\Phi_i^{-1} B_i}{\sum_j \Phi_j^{-1} B_j}, \tag{2.4.2}$$

where Φ_i denotes the elasticity of expenditure on commodity i with respect to quantity of commodity i. In the corresponding cost minimization problem, the elasticity of expenditure with respect to utility, Φ, is given by:

$$\Phi^{-1} = \sum_i \Phi_i^{-1} B_i. \tag{2.4.3}$$

The proof is a straightforward exercise in constrained optimisation.

Note that with expenditures proportional to quantities, the Φ_i are unity, so the equation reduces to the standard Cobb-Douglas fixed-shares equation. In general however the expenditure shares are variable.

Now the government consumption demand system is Cobb-Douglas, so it is homothetic, so we can cardinalize utility from government consumption so that $\Phi_G \equiv 1$. Also saving is a single commodity, so $\Phi_S \equiv 1$. Applying then proposition 2 to the GTAP demand system, we have

$$\frac{X_P}{X} = \left(\frac{\Phi_P}{\Phi}\right)^{-1} B_P,$$
$$\frac{X_G}{X} = \Phi B_G,$$
$$\frac{X_S}{X} = \Phi B_S,$$

or, in percentage change form,

$$\boxed{x_P - x = -(\phi_P - \phi)} \tag{2.4.4}$$

$$\boxed{x_G - x = \phi} \tag{2.4.5}$$

$$\boxed{x_S - x = \phi} \tag{2.4.6}$$

For percentage change in the utility elasticity of income, ϕ, we have

$$\begin{aligned}
\phi &= \sum_i \frac{\Phi_i^{-1} B_i}{\sum_j \Phi_j^{-1} B_j} \phi_i && \text{differentiating (2.4.3),} \\
&= \sum_i \frac{X_i}{X} \phi_i && \text{substituting from (2.4.2),} \\
&= \sum_i S_i \phi_i && \text{putting } S_i = X_i/X, \\
&= S_P \phi_P + S_G \phi_G + S_S \phi_S.
\end{aligned}$$

Since $\phi_G = \phi_S = 0$, this reduces to

$$\boxed{\phi = S_P \phi_P} \tag{2.4.7}$$

As we see from these equations, the utility elasticity of income, Φ, is a weighted average of the lower-level utility elasticities Φ_P, Φ_G, and Φ_S. Since Φ_G and Φ_S are fixed, changes in Φ depend only on changes in the utility elasticity of private consumption expenditure, Φ_P. An increase in Φ_P, a shift so to speak towards decreasing returns from private consumption, leads to a budget reallocation away from private consumption toward government consumption and saving.

We now develop an equation for changes in Φ_P. As shown by Hanoch [3], with the CDE form for the private consumption demand system, the utility elasticity is a weighted average of the expansion parameters:

$$\Phi_P = \sum_i S_i^P E_i. \tag{2.4.8}$$

Differentiating, we obtain

$$\phi_P = \sum_i S_{Ei} s_{Pi}$$

where S_{Ei} denotes the expansion-parameter-weighted budget share of commodity i,

$$S_{Ei} = \frac{S_i^P E_i}{\sum_j S_j^P E_j} = \frac{S_i^P E_i}{\Phi_P}. \tag{2.4.9}$$

Then writing p_{Pi} for the price of commodity i in private consumption, and u_{Pi} for *per capita* private consumption of commodity i, we obtain

$$\boxed{\phi_P = \sum_i S_{Ei}(p_{Pi} + u_{Pi} - x_P)} \tag{2.4.10}$$

We see from this equation that shifts in private expenditure allocation toward commodities with high expansion parameters E_i tend to be associated with increases in the private expenditure utility elasticity, while shifts towards commodities with low expansion parameters tend to be associated with decreases.

For aggregate utility we use the general result:

Proposition 3 *For the upper level of a weakly separable demand system,*

$$\textit{max } U(U_1, \ldots, U_G) \textit{ subject to } \sum_i E(\mathbf{P_i}, U_i) = X,$$

where $E(\mathbf{P_i}, U_i)$ denotes the expenditure function for the i'th lower-level demand system, we have

$$x = p + \Phi u,$$

where p is an expenditure-share-weighted index of commodity group price indices, $p = \sum_i S_i p_i$, where S_i denotes the share of expenditure on group i in total expenditure, $S_i = X_i/X$, and p_i is an expenditure-weighted index of prices of commodities in group i, $p_i = \sum_j S^i_j p_{ij}$, where S^i_j denotes the share of commodity j from group i in total expenditure on group i, $S^i_j = X_{ij}/X_i$, where X_{ij} denotes expenditure on commodity j from group i, and p_{ij} denotes the price of commodity j from group i.

Proof. Define the Lagrangean

$$\mathcal{L} = U(U_1, \ldots, U_G) - \Lambda \left(\sum_i E(\mathbf{P_i}, U_i) - X \right). \tag{2.4.11}$$

Then the elasticity of utility with respect to income,

$$\begin{aligned} \frac{\partial \log U}{\partial \log X} &= \frac{X}{U} \frac{\partial \mathcal{L}}{\partial X} && \text{by the envelope theorem} \\ &= \frac{X}{U} \Lambda && \text{differentiating (2.4.11)} \\ &= \Phi^{-1}, \end{aligned}$$

and the elasticity of utility with respect to the price of the j'th commodity in the i'th commodity group, that is, with respect to the j'th component of $\mathbf{P_i}$,

$$\begin{aligned} \frac{\partial \log U}{\partial \log P_{ij}} &= \frac{P_{ij}}{U} \frac{\partial \mathcal{L}}{\partial P_{ij}} && \text{by the envelope theorem} \\ &= -\frac{P_{ij}}{U} \Lambda \frac{\partial X_i}{\partial P_{ij}} && \text{differentiating (2.4.11)} \\ &= -\Lambda \frac{P_{ij} Q_{ij}}{U} && \text{by Shephard's lemma} \\ &= -\frac{X}{U} \Lambda \frac{P_{ij} Q_{ij}}{X} \\ &= -\Phi^{-1} S_{ij}, \end{aligned}$$

where Q_{ij} denotes consumption, and S_{ij} the share in total expenditure, of commodity j in commodity group i (for the envelope theorem see e.g. Varian [7]). Then totally differentiating the indirect utility function, we have

$$\begin{aligned} u &= \sum_i \sum_j \frac{\partial \log U}{\partial \log P_{ij}} p_{ij} + \frac{\partial \log U}{\partial \log X} x \\ &= -\Phi^{-1} \sum_i \sum_j S_{ij} p_{ij} + \Phi^{-1} x \\ &= \Phi^{-1}(x - p), \end{aligned}$$

where p is the expenditure-share-weighted index of commodity group price indices,

$$\begin{aligned} p &= \sum_i \sum_j S_{ij} p_{ij} \\ &= \sum_i \sum_j \frac{X_{ij}}{X} p_{ij} = \sum_i \frac{X_i}{X} \sum_j \frac{X_{ij}}{X_i} p_{ij} = \sum_i S_i \sum_j S_j^i p_{ij} \\ &= \sum_i S_i p_i, \end{aligned}$$

as in the statement of the proposition. Solving for x, we obtain

$$x = p + \Phi u,$$

as was to be shown.■

Moving from the general formulation of proposition 3 to the specific case of the GTAP upper-level demand system, we copy the utility equation verbatim:

$$\boxed{x = p + \Phi u} \tag{2.4.12}$$

but write the disposition price index equation in the more specific form

$$\boxed{p = S_P p_P + S_G p_G + S_S p_S} \tag{2.4.13}$$

where p_P and p_G denote expenditure-weighted price indices for private and government consumption, and p_S denotes the price of saving.

2.5 Defects in the old treatment: further findings

Having identified an error in the derivation of the old theory (subsection 2.2), and revised the theory to remove that defect (subsection 2.4), we now compare the results of the revised theory with the original.

From equations (2.4.4)–(2.4.6), we see that under the revised theory, the upper-level income disposition shares are not in general fixed. They are fixed in the special case $\phi = \phi_P = 0$; from equation (2.4.7), this condition reduces to $\phi_P = 0$; from equation (2.4.8), this is satisfied if fixed private consumption expenditure shares S_i^P or uniform expansion parameters $E_i \equiv E$; that is, in the special case of a homothetic system. In general however, the old top-level demand equations, which assume fixed income disposition shares, are in error.

For utility, the old and new treatments use rather different approaches, so we cannot directly compare the two equations. Instead we derive a new utility equation consistent with the new theory but similar in approach to the old equation. We follow the old derivation as far as equation (2.1.12),

$$u = B_P u_P + B_G(q_G - n) + B_S(q_S - n).$$

Then, instead of the old $S_i = B_i/B$ (paraphrasing equations (2.1.4)–(2.1.6)), we use the new equation (2.4.2), $S_i = \Phi_i^{-1} B_i / \sum_j \Phi_j^{-1} B_j$. From this and equation (2.4.3), $\Phi^{-1} = \sum_i \Phi_i^{-1} B_i$, we obtain

$$B_i = \frac{\Phi_i}{\Phi} S_i,$$

for $i = P, G, S$. Substituting into equation (2.1.12), and setting $\Phi_G = \Phi_S = 1$, we obtain

$$u = \Phi^{-1}[\Phi_P S_P u_P + S_G(q_G - n) + S_S(q_S - n)].$$

Comparing this with the old utility equation (2.1.13),

$$\begin{aligned} Yu &= Y_P u_P + Y_G(q_G - n) + Y_S(q_S - n), \\ \Leftrightarrow \qquad u &= S_P u_P + S_G(q_G - n) + S_S(q_S - n), \end{aligned} \tag{2.5.1}$$

we note that the old computation is invalid in general, but valid in the special case $\Phi = 1,\ \Phi_P = 1$. As we now show, standard GTAP data bases fall within the special case.

Proposition 4 *Under the old treatment, the utility elasticity of income is equal to one if and only if the share-weighted sum of the CDE expansion parameters is equal to one.*

Proof. We have, from general theory, and from the treatment of saving and government consumption,

$$\begin{aligned} u_S &= x_S - p_S, \\ u_G &= x_G - p_G, \\ u_P &= \Phi_P^{-1}(x_P - p_P). \end{aligned}$$

With the fixed-expenditure-shares upper-level demand equations in the old system, this simplifies to

$$\begin{aligned} u_S &= x - p_S, \\ u_G &= x - p_G, \\ u_P &= \Phi_P^{-1}(x - p_P). \end{aligned}$$

Then recalling equation (2.5.1), we have

$$\begin{aligned} u &= S_P u_P + S_G(q_G - n) + S_S(q_S - n) \\ &= S_P u_P + S_G u_G + S_S u_S \\ &= \Phi_P^{-1} S_P(x - p_P) + S_G(x - p_G) + S_S(x - p_S). \end{aligned}$$

Recalling equation (2.4.12), we have

$$u = \Phi^{-1}(x - p),$$

where

$$p = S_P p_P + S_G p_G + S_S p_S.$$

we find (ignoring the pathological case $S_P = 0$) that the two equations for u are consistent if and only if $\Phi_P = 1$. If however we ignore the effects of price changes, but consider income changes only, we find that the equations are consistent provided

$$\Phi^{-1} = S_P \Phi_P^{-1} + S_G + S_S.$$

We interpret the value of Φ from this equation as the value implicit in the model. Then (again assuming $S_P \neq 0$) $\Phi = 1$ if and only if $\Phi_P = 1$. But by equation (2.4.8), Φ_P is the expenditure-share-weighted sum of the CDE expansion parameters. So $\Phi = 1$ if and only if the expenditure-share-weighted sum of the CDE expansion parameters is equal to one; as was to be shown.■

In constructing standard GTAP data bases, we have normalized the expansion parameters so that their expenditure-share-weighted sum is indeed equal to one. Then, from equation (2.4.8) and proposition 4, both the utility elasticity of private consumption expenditure and the utility elasticity of income are equal to one; so the old utility equation is valid locally. Since however normalization is not a theoretical requirement of the CDE, users may legitimately construct data bases with non-normalized parameters; and with those data bases, the utility equation is invalid. Furthermore, in multi-step simulations, initially normalized expansion parameters do not generally remain normalized; so even with initially normalized parameters, the utility equation is not exact.

The old utility equation (2.5.1) then is exactly accurate in Johansen simulations with data bases in which $\Phi = \Phi_P = 1$ (including standard GTAP data bases); accurate to first order in multi-step simulations with data bases in which $\Phi = \Phi_P = 1$; and inaccurate otherwise.

We note also that in GTAP simulations with the old treatment, the results for utility are inaccurate even in Johansen simulations with standard GTAP data bases. This is because, although the utility equation itself is exact, the upper-level demand equations are inaccurate. In practice however, with standard data bases, errors in the utility results are likely to be small (see further section 5).

2.6 Possible remedies

There are several different approaches we might take to remedy the defects of the old treatment.

1. We might retain the basic premises of the old treatment, in particular, the CDE form for the private consumption demand system, while correcting the errors in the derived equations, adopting the revised theory expounded above (subsection 2.4).

2. We might adopt a new functional form for the private consumption demand system, that would allow us to retain fixed budget shares in the upper-level system.

3. We might abandon the concept of a upper-level demand system. Rather than representing the allocation of regional income as optimizing behavior by a fictitious regional household, we might simply impose some arbitrary rule. There would not necessarily be a concept of regional welfare, but instead a purely descriptive treatment of macroeconomic behavior. This might be a simple rule such as the fixed shares rule, or some more complex empirically motivated treatment.

Option (1) has the advantage of maximizing theoretical consistency with the old treatment. Its disadvantage is that the upper-level demand equations become more complex, so that the upper-level budget shares are no longer fixed. Options (2) and (3) let us keep the fixed budget shares property, but require changes in the basic theory. Option (3) also entails some fundamental change in the welfare analysis.

In this paper (subsections 2.8– 2.16) we concentrate on option (1). It seems reasonable to develop the implications of the current basic theory before entertaining proposals to modify it. First however (subsection 2.7) we verify that (2) is, as implied above, a feasible option.

2.7 A digression on alternative private consumption demand systems

As shown above (subsection 2.3), unless we accept a homothetic private consumption demand system, we must accept some changes to the upper-level system. On the other hand, while not concerned to retain all aspects of the current upper-level system, we would like to retain at least the fixed shares property. In this section we investigate whether we can find a new form for the private consumption demand system such as to preserve the fixed shares property while perhaps affecting other aspects of the upper-level system.

Recalling equation (2.4.2),

$$\frac{X_i}{X} = \frac{\Phi_i^{-1} B_i}{\sum_j \Phi_j^{-1} B_j},$$

we see that even when the elasticities Φ are not all equal to one, the budget shares are constant provided that the elasticities are constant. This seems a hopeful notion: with constant elasticities, we change some aspects of the upper-level system but retain the fixed budget shares. As it turns out however, this approach imposes unacceptable restrictions on the form of the lower-level systems.

Proposition 5 *In any demand system, if the utility elasticity of expenditure is constant, the system is homothetic.*

Proof. Let $\bar{U}$ denote some arbitrary elasticity level, and Φ the constant utility elasticity. If the utility elasticity of expenditure is constant, then for any utility level U, $E(\mathbf{P}, U) = (U/\bar{U})^{\Phi} E(\mathbf{P}, \bar{U})$. But then we can write, for all $\mathbf{P}$, U, $E(\mathbf{P}, U) = \Pi(\mathbf{P})(U/\bar{U})^{\Phi}$, where $\Pi(\mathbf{P}) = E(\mathbf{P}, \bar{U})$. So, by proposition 1, the system is homothetic.■

Since homotheticity is empirically unacceptable, this idea does not help us find an acceptable form for the private consumption demand system.

We may also attempt to use the Gorman [2] conditions for two stage budgeting to find a functional form for the private consumption demand system that lets us preserve the upper-level demand system. This is a somewhat subtle strategy. We have seen above (subsection 2.3) that there is no non-homothetic private consumption demand system that, in conjunction with a Cobb-Douglas upper-level utility function $U_\bullet$, leads to fixed upper-level budget shares. There might yet however be a non-homothetic private consumption demand system that, in conjunction with a upper-level utility function not of the Cobb-Douglas form, leads to a Cobb-Douglas upper-level utility index $U^*_\bullet$. This is in fact feasible.

Of the two alternative conditions in [2], one entails homothetic lower-level demand systems, which is unacceptable. The other condition however does allow an acceptable solution.

Proposition 6 *In a two-level demand system with an upper-level additive utility function* $U_\bullet(U_1, \ldots, U_G) = \sum_{i=1}^G U_i$ *and lower-level indirect utility functions*

$$\Psi_i(\mathbf{P_i}, X_i) = B_i \log \frac{X_i}{M_i(\mathbf{P_i})} + A_i(\mathbf{P_i}), \tag{2.7.1}$$

the upper-level expenditure shares are fixed.

Proof. In consumer equilibrium, the group expenditure levels X_i solve the problem

> Find X_i to maximize $\sum_i \Psi_i(\mathbf{P_i}, X_i)$ such that $\sum_i X_i = X$;

that is, with the specified form for the lower-level indirect utility functions Ψ_i,

> Find X_i to maximize
>
> $$\sum_i B_i \log\left(\frac{X_i}{M_i(\mathbf{P_i})}\right) + \sum_i A_i(\mathbf{P_i})$$
>
> such that $\sum_i X_i = X$.

Since the functions A_i do not involve group expenditure X_i, this is equivalent to

> Find X_i to maximize $\sum_i B_i \log(X_i/M_i(\mathbf{P_i}))$ such that $\sum_i X_i = X$;

or, putting $U^*_i = X_i/M_i(\mathbf{P_i})$, $P_i = M_i(\mathbf{P_i})$,

> Find U^*_i to maximize $\sum_i B_i \log U^*_i$ such that $\sum_i P_i U^*_i = X$.

This has the standard Cobb-Douglas solution

$$U_i^* = \frac{B_i}{\sum_j B_j} \frac{X}{P_i},$$

so the expenditure shares

$$\frac{X_i}{X} = \frac{P_i U_i^*}{X} = \frac{B_i}{\sum_j B_j},$$

so the expenditure shares are fixed, as was to be shown.■

The functional form (2.7.1) covers both (with zero A_G) the Cobb-Douglas demand system used in GTAP for government consumption, and (with non-zero A_P) a reasonably extensive class of non-homothetic private consumption demand systems. So with the demand system of proposition 6, we can preserve the Cobb-Douglas government consumption system and the upper-level fixed shares, by changing the upper-level utility function and the private consumption demand system.

2.8 A new treatment

We now develop a new treatment for the upper-level demand system. As discussed above (subsection 2.6), we correct errors in the old theory without changing its basic premises. In particular, we retain the CDE form for the demand system for private consumption.

We do change one minor feature of the old framework: we redefine utility from government consumption, `ug`, as a *per capita* utility, so that it depends on *per capita* rather than total government consumption. Since saving and utility from private consumption are already *per capita* variables, this change allows us to treat the entire regional household demand system as the demand system of a representative regional household, rather than as a conglomeration of demand systems of different households (subsection 2.2).

To allow for exogenous shifts in the upper-level budget allocation, we treat the Cobb-Douglas distribution parameters B_i as variables. This allows the model to simulate exogenous budget shifts within the demand system, rather than (as with the old treatment) by overriding the demand system. With this addition, we use the revised theory derived above (subsection 2.4).

We modify the module structure within the GTAP model source code, to bring within the regional household module all equations derived from the upper level of the final demand system, rather than leaving them scattered across the regional household, government household, and investment and saving modules.

2.9 Shared variables

To implement the revised system, we first define some new cross-module variables. In the new theory, the private consumption and regional household modules share the levels coefficient Φ_P for the elasticity of private consumption expenditure with respect to utility from private consumption:

```
Coefficient (all,r,REG)
    UELASPRIV(r)
    #elasticity of cost wrt utility from private consumption#;
```

the corresponding percentage variable ϕ_P:

```
Variable (all,r,REG)
    uepriv(r)
    #elasticity of cost wrt utility from private consumption#;
```

and p_P, the private consumption price index:

```
Variable (all,r,REG)
    ppriv(r)
    #price index for private consumption expenditure in region r#;
```

The government consumption and regional household modules share y_G, government consumption expenditure:

```
Variable (all,r,REG)
    yg(r)
    #regional government consumption expenditure, in region r#;
```

At the same time, some variables previously shared between modules now become localised to individual modules. Utility from government consumption, `ug`, the prices of composite commodities in government consumption, `pg`, and quantities of composite commodities consumed by government, `qg`, become local to the government consumption module. Utility from private consumption, `up`, becomes local to the private consumption module.

2.10 Government consumption

Following the redefinition of utility U_G from government consumption as a *per capita* variable (subsection 2.8), we make the consequential changes in the government consumption module. Specifically, we revise the government consumption utility equation:

```
Equation GOVU
# utility from government consumption in r #
(all,r,REG)
    yg(r) - pop(r) = pgov(r) + ug(r);
```

and the government consumption demand equation:

```
Equation GOVDMNDS
# government consumption demands for composite commodities (HT 41) #
(all,i,TRAD_COMM)(all,r,REG)
    qg(i,r) - pop(r) = ug(r) - [pg(i,r) - pgov(r)];
```

Besides making these substantive changes, we remove all references to the government household from comments and labels in the source code.

2.11 Utility from private consumption

Within the private consumption module, we need new code for the utility elasticity of private consumption expenditure, the private consumption price index, and utility from private consumption.

To implement, we compute the level of the utility elasticity Φ_P according to equation (2.4.8):

```
Formula (all,r,REG)
    UELASPRIV(r) = sum{i,TRAD_COMM, CONSHR(i,r)*INCPAR(i,r)};
```

the expansion-parameter-weighted budget shares S_{Ei} according to equation (2.4.9):

```
Coefficient (all,i,TRAD_COMM)(all,r,REG)
    XWCONSHR(i,r)
    #expansion-parameter-weighted consumption share#;
Formula (all,i,TRAD_COMM)(all,r,REG)
    XWCONSHR(i,r) = CONSHR(i,r)*INCPAR(i,r)/UELASPRIV(r);
```

and percentage change in the utility elasticity ϕ_P according to equation (2.4.10):

```
Equation UTILELASPRIV
#elasticity of expenditure wrt utility from private consumption#
(all,r,REG)
    uepriv(r)
  = sum{i,TRAD_COMM, XWCONSHR(i,r)*[pp(i,r) + qp(i,r) - yp(r)]};
```

For utility from private consumption, we replace the perfectly satisfactory computation in the old code,

```
Equation PRIVATEU
# computation of utility from private consumption in r (HT 45) #
(all,r,REG)
    yp(r)
        = sum(i,TRAD_COMM, (CONSHR(i,r) * pp(i,r)))
        + sum(i,TRAD_COMM, (CONSHR(i,r) * INCPAR(i,r))) * up(r)
        + pop(r)
        ;
```

with a more readily interpretable computation based on the following general proposition:

Proposition 7 *For a demand system,*

$$\max U(Q_1, \ldots, Q_I) \text{ subject to } \sum_i P_i Q_i = X,$$

we have

$$x = p + \Phi u,$$

where p is an expenditure-share-weighted index of commodity prices, $p = \sum_i p_i$.

Proof. This is a special case of proposition 3, where the lower-level demand systems each cover just one commodity and the subutilities U_i are just the commodity consumption quantities Q_i.■

Applying proposition 7 to utility from private consumption, we have

$$y_P - n = p_P + \Phi_P u_P, \tag{2.11.1}$$

where the price index for private consumption,

$$p_P = \sum_i S_i^P p_{Pi}, \tag{2.11.2}$$

We compute the private consumption price index p_P according to equation (2.11.2):

```
Equation PHHLDINDEX
# price index for private consumption expenditure #
(all,r,REG)
    ppriv(r) = sum{i,TRAD_COMM, CONSHR(i,r)*pp(i,r)};
```

and utility from private consumption according to equation (2.11.1):

```
Equation PRIVATEU
# computation of utility from private consumption in r (HT 45) #
(all,r,REG)
    yp(r) - pop(r) = ppriv(r) + UELASPRIV(r)*up(r);
```

2.12 Regional household preliminaries

Within the regional household module we revise the submodules for regional household demands (subsection 2.13) and aggregate utility (subsection 2.14). We compute at the outset some coefficients common to both submodules, the upper-level shares S_i in regional income, $S_i = X_i/X$:

```
Coefficient (all,r,REG)
    XSHRPRIV(r) #private expenditure share in regional income#;
Formula (all,r,REG)
    XSHRPRIV(r) = PRIVEXP(r)/INCOME(r);

Coefficient (all,r,REG)
    XSHRGOV(r) #government expenditure share in regional income#;
Formula (all,r,REG)
    XSHRGOV(r) = GOVEXP(r)/INCOME(r);

Coefficient (all,r,REG)
    XSHRSAVE(r) #saving share in regional income#;
Formula (all,r,REG)
    XSHRSAVE(r) = SAVE(r)/INCOME(r);
```

We also declare some common variables: the distribution parameters b_i from the top-level demand equation:

```
Variable (all,r,REG)
    dppriv(r) #private consumption distribution parameter#;
Variable (all,r,REG)
    dpgov(r) #government consumption distribution parameter#;
Variable (all,r,REG)
    dpsave(r) #saving distribution parameter#;
```

and ϕ, the utility elasticity of income:

```
Variable (all,r,REG)
    uelas(r) #elasticity of cost of utility wrt utility#;
```

2.13 Regional household demands

We extend the revised theory (subsection 2.4) to treat the Cobb-Douglas distribution parameters of the upper-level demand system as variables in the simultaneous equation system. For the demand equations, we extend equations (2.4.4)–(2.4.6), obtaining

$$x_P - x = -(\phi_P - \phi) + b_P, \tag{2.13.1}$$

$$x_G - x = \phi + b_G, \tag{2.13.2}$$

$$x_S - x = \phi + b_S. \tag{2.13.3}$$

For the utility elasticity of income, ϕ, we extend equation (2.4.7), obtaining

$$\phi = S_P \phi_P - b_{AV}, \tag{2.13.4}$$

where b_{AV} denotes a weighted average of the distribution parameters,

$$b_{AV} = \sum_i S_i b_i. \tag{2.13.5}$$

To implement this, we first declare the distribution parameters b_i:

```
Variable (all,r,REG)
    dppriv(r) #private consumption distribution parameter#;
Variable (all,r,REG)
    dpgov(r) #government consumption distribution parameter#;
Variable (all,r,REG)
    dpsave(r) #saving distribution parameter#;
```

We then compute the weighted average of the distribution parameters according to equation (2.13.5):

```
Variable (all,r,REG)
    dpav(r) #average distribution parameter shift, for EV calc.#;
Equation DPARAV #average distribution parameter shift#
    (all,r,REG)
    dpav(r)
  =   XSHRPRIV(r)*dppriv(r)
    + XSHRGOV(r)*dpgov(r)
    + XSHRSAVE(r)*dpsave(r)
    ;
```

the utility elasticity of income according to equation (2.13.4):

```
Equation UTILITELASTIC #elasticity of cost of utility wrt utility#
    (all,r,REG)
    uelas(r) = XSHRPRIV(r)*uepriv(r) - dpav(r);
```

and regional household demands according to equations (2.13.1)–(2.13.3):

```
Equation PRIVCONSEXP #private consumption expenditure# (all,r,REG)
    yp(r) - y(r) = -[uepriv(r) - uelas(r)] + dppriv(r);

Equation GOVCONSEXP #government consumption expenditure# (all,r,REG)
    yg(r) - y(r) = uelas(r) + dpgov(r);

Equation SAVING #saving# (all,r,REG)
    psave(r) + qsave(r) - y(r) = uelas(r) + dpsave(r);
```

2.14 Regional household utility

Now we compute utility for the regional household. Recalling the levels equation (2.4.1),

$$U = C \prod_i U_i^{B_i},$$

we extend the differential equation (2.4.12) to treat the scaling factor C and the distribution parameters B_i as variable, obtaining

$$u = c + \sum_i B_i (\log U_i) b_i + \Phi^{-1}(x - p). \tag{2.14.1}$$

We remark that the initial settings of $\log U_i$ are arbitrary, in that they are not constrained by the observed state of the economy as recorded in the data base, and do not affect the positive properties of the demand system. They affect only the sensitivity of utility to changes in preferences. Once the initial settings have been made however, theory dictates how the coefficients should be updated. By adjusting the settings of $\log U_i$, we can make utility increasing in the distribution parameters, decreasing, or locally invariant. We can also make it increasing with respect to some of the distribution parameters and decreasing with respect to others.

The requirements for implementing distribution terms in the equation are somewhat onerous, in that we need to store and update both the distribution parameters B_i and the quantities U_i—even though these are not required for any positive variables. Given all this, and the doubtful meaningfulness of utility comparisons in the presence of preference changes, it may seem hardly worthwhile incorporating the distribution parameters into the utility equation. Yet we attach some importance to it. Some important macro closures involve exogenizing the balance of trade and endogenizing a distributional variable. It would be a great inconvenience when using these closures to forego results for utility and equivalent variation and the welfare decomposition. Moreover it seems that most of the welfare analysis should be just as meaningful with an exogenous as with an endogenous trade balance.

Since we must have the distributional parameters but do not welcome their welfare effects, we do what we can to minimize them. We choose initial parameter values so that, in small change simulations, changes in the distributional parameters do not affect utility (subsection 2.16). And we provide, in connection with the measurement of equivalent variation, a mechanism for minimizing the welfare effects of distributional parameter changes in large-change simulations (subsection 3.8).

To implement equation (2.14.1), we first declare as percentage change variables utility u:

```
Variable (all,r,REG)
    u(r) #per capita utility from aggregate hhld expend., in region r#;
```

and the constant c in the utility function:

```
Variable (all,r,REG)
    au(r) #input-neutral shift in utility function#;
```

We need next the levels values of the distributional parameters B_i. From equations (2.4.2) and (2.4.3), we find that we can calculate them as

$$B_i = \frac{\Phi_i S_i}{\Phi}, \tag{2.14.2}$$

given the levels value of the utility elasticity Φ. The theory however does not determine the levels value of Φ. We could store Φ in the data base, but it is slightly more convenient to store instead the sum of the distribution parameters, $B = \sum_i B_i$; since when the utility elasticity of private consumption expenditure, Φ_P, is non-unitary, it is more natural to take B than Φ as unitary, leaving the other coefficient to take a non-obvious calculated value. From equation (2.14.2), we obtain the formula giving Φ in terms of given B:

$$\Phi = \frac{\sum_i S_i \Phi_i}{B}. \tag{2.14.3}$$

To calculate the sum B of the distribution parameter in updated databases, we use the corresponding percentage change variable b. We declare this variable:

```
Variable (all,r,REG)
    dpsum(r) #sum of the distribution parameters#;
```

and define the corresponding levels coefficient:

```
Coefficient (all,r,REG)
    DPARSUM(r) #sum of distribution parameters#;
Read
    DPARSUM from file GTAPDATA header "DPS";
Update (all,r,REG)
    DPARSUM(r) = dpsum(r);
```

This lets us define the level Φ of the utility elasticity of expenditure, according to equation (2.14.3):

```
Coefficient (all,r,REG)
    UTILELAS(r) #elasticity of cost of utility wrt utility#;
Formula (all,r,REG)
    UTILELAS(r)
  = [UELASPRIV(r)*XSHRPRIV(r) + XSHRGOV(r) + XSHRSAVE(r)]/DPARSUM(r);
```

We define the levels coefficients B_i for the distribution parameters using equation (2.14.2):

```
Coefficient (all,r,REG)
    DPARPRIV(r) #private consumption distribution parameter#;
Formula (all,r,REG)
    DPARPRIV(r) = UELASPRIV(r)*XSHRPRIV(r)/UTILELAS(r);

Coefficient (all,r,REG)
    DPARGOV(r) #government consumption distribution parameter#;
Formula (all,r,REG)
    DPARGOV(r) = XSHRGOV(r)/UTILELAS(r);

Coefficient (all,r,REG)
    DPARSAVE(r) #saving distribution parameter#;
Formula (all,r,REG)
    DPARSAVE(r) = XSHRSAVE(r)/UTILELAS(r);
```

We define also the levels coefficients U_i for the goods in the top-level utility function:

```
Coefficient (all,r,REG)
    UTILPRIV(r) #utility from private consumption#;
Read
    UTILPRIV from file GTAPDATA header "UP";
Update (all,r,REG)
    UTILPRIV(r) = up(r);

Coefficient (all,r,REG)
    UTILGOV(r) #utility from government consumption#;
```

```
Read
    UTILGOV from file GTAPDATA header "UG";
Update (all,r,REG)
    UTILGOV(r) = ug(r);

Coefficient (all,r,REG)
    UTILSAVE(r) #utility from saving#;
Read
    UTILSAVE from file GTAPDATA header "US";
Update (change) (all,r,REG)
    UTILSAVE(r) = 100*[qsave(r) - pop(r)]*UTILSAVE(r);
```

We compute the outlays price index p according to equation (2.4.13):

```
Variable (all,r,REG)
    p(r) #price index for disposition of income by regional household#;
Equation PRICEINDEXREG
    #price index for disposition of income by regional household#
    (all,r,REG)
    p(r)
  =   XSHRPRIV(r)*ppriv(r)
    + XSHRGOV(r)*pgov(r)
    + XSHRSAVE(r)*psave(r)
    ;
```

After all these preliminaries, we compute regional household utility u, according to equation (2.4.12):

```
Equation UTILITY #regional household utility# (all,r,REG)
  u(r) = au(r)
    + DPARPRIV(r)*loge(UTILPRIV(r))*dppriv(r)
    + DPARGOV(r)*loge(UTILGOV(r))*dpgov(r)
    + DPARSAVE(r)*loge(UTILSAVE(r))*dpsave(r)
    + [1.0/UTILELAS(r)]*[y(r) - pop(r) - p(r)];
```

One task remains, to determine the variable `dpsum` used to update the coefficient `DPARSUM`:

```
Equation DISTPARSUM #sum of the distribution parameters# (all,r,REG)
    DPARSUM(r)*dpsum(r)
  = DPARPRIV(r)*dppriv(r) + DPARGOV(r)*dpgov(r) + DPARSAVE(r)*dpsave(r);
```

2.15 Shifting income allocation without affecting the utility elasticity

In the new treatment, changes in the distribution parameters do not affect the current level of utility (utility is compensated as described in subsection 2.14), but they do generally affect the utility elasticity of income. This in turn affects the relation between income y and utility u in simulation results. It would be nice to be able to modify the upper-level income distribution without changing the utility elasticity of income. To that end we provide the distribution parameters with item-specific and generic shift variables.

From equation (2.13.4), we see that the distribution parameters affect the utility elasticity through the average distribution parameter b_{AV}. So to change the distribution parameters without affecting the marginal utility of income, the user makes b_{AV} exogenously zero. The user may swap b_{AV} either with one of the b_i or with a generic distribution parameter scaling factor. To support this, we write each distribution parameter as the product of a specific and the generic scaling factor:

```
Variable (all,r,REG)
    dpfpriv(r) #private-consumption-specific distparam shift#;
Variable (all,r,REG)
    dpfgov(r) #government-consumption-specific distparam shift#;
Variable (all,r,REG)
    dpfsave(r) #saving-specific distparam shift#;
Variable (all,r,REG)
    dpshift(r) #generic distparam shift#;

Equation DISTPARPRIV #private consumption distribution parameter# (all,r,REG)
    dppriv(r) = dpfpriv(r) + dpshift(r);
Equation DISTPARGOV #government consumption distribution parameter# (all,r,REG)
    dpgov(r) = dpfgov(r) + dpshift(r);
Equation DISTPARSAVE #saving distribution parameter# (all,r,REG)
    dpsave(r) = dpfsave(r) + dpshift(r);
```

If no exogenous change in distribution is desired, the user may make the distribution parameters endogenous, the average of the distribution parameters endogenous, and the distribution parameter shift variables exogenous. To shift say the saving share in income without changing the utility elasticity,

- make exogenous:
  - the saving distribution parameter `dpsave`,
  - the average distribution parameter `dpav`,
  - the private consumption distribution parameter shift `dpfpriv`, and
  - the government consumption distribution parameter shift `dpfgov`;

- make endogenous:
  - the private consumption distribution parameter `dppriv`,
  - the government consumption distribution parameter `dpgov`,
  - the saving distribution parameter shift `dpfsave`, and
  - the distribution parameter generic shift `dpshift`; and
- shock the saving distribution parameter `dpsave` as desired.

Because the marginal utility of income depends on the budget shares S_i, this approach keeps it fixed only locally, not globally. Furthermore, it introduces a path dependency into the model, affecting `dpshift` and the endogenous distribution parameter variables (in the example shown above, `dppriv` and `dpgov`). If the user considers path dependency objectionable, she may prefer to leave `dpshift` exogenous, and apply to it a shock calculated *ex ante* outside the model.

2.16 Changes to the data file

As described in subsection 2.14, we read a new coefficient `DPARSUM` from the data file. To do this we need a new data file array `DPS`, with dimension `REG`. The new array records, for each region, the sum of the distribution parameters.

The setting of this parameter has no effect on the positive variables in the model, nor on the equivalent variation, but through the top-level utility elasticity `UTILELAS` it does affect regional utility `u`. We set it initially at 1 in each region; changes in the distribution parameters `dppriv`, `dpgov`, and `dpsave` may affect its value in updated data bases.

In standard data bases, with both `UTILELASPRIV` and `DPSUM` set equal to 1, the utility elasticity `UTILELAS` of generalized expenditure is equal to 1. This means that initially, a one per cent change in regional income translates into a one per cent change in regional utility.

We also set values for three region-dimension arrays representing levels for the commodities in the top level of the regional demand system: utility from private consumption, `UP`; utility from government consumption, `UG`; and saving, `Q`. We set these all to zero to ensure that with the standard data base, changes in the distributional parameters have no first-order effect on utility (subsection 2.14).

3 Equivalent variation

We derive the old treatment (subsection 3.1), assess its defects (subsection 3.2), develop a new treatment (subsection 3.3), and implement it (subsections 3.5–3.10).

By definition, the equivalent variation (EV),

$$EV = Y_{EV} - \bar{Y},$$

where Y_{EV} denotes regional income required to achieve current utility at initial prices, and $\bar{Y}$ denotes initial regional income. Differentiating, we obtain:

$$d\,EV = \frac{1}{100} Y_{EV} y_{EV}. \tag{3.0.1}$$

This equation provides a starting point for both the old and new treatments.

3.1 The old treatment

In the old treatment, EV is computed according to the equation

```
Equation EVREG
# regional EV, the money metric welfare change (HT 67) #
(all,r,REG)
    EV(r)
  = [REGEXP(r)/100]*[URATIO(r)*POPRATIO(r)]*[u(r) + pop(r)];
```

In mathematical notation, we may write this as

$$d\,EV = \frac{1}{100} \bar{Y} U_R N_R (n + u), \tag{3.1.1}$$

where $U_R = U/\bar{U}$ is the ratio of current to initial utility, $N_R = N/\bar{N}$ the ratio of current to initial population, and n the percentage change in population. So far as we are aware, no derivation of this equation is available in the original GTAP documentation (Hertel and Tsigas [5]) or earlier GTAP technical papers or working papers. We now provide one, in order to explore the conditions under which the equation is valid.

Proposition 8 *Equation* (3.1.1) *is a valid first-order approximation for small changes in* U*, provided that initially the utility elasticity of income,* Φ*, is equal to one.*

Proof. Recall equation (3.0.1),

$$d\,EV = \frac{1}{100} Y_{EV} y_{EV}.$$

Now

$$y_{EV} = n + x_{EV},$$

where x_{EV} denotes percentage change in *per capita* expenditure required to achieve current utility at initial prices. Also, adapting equation (2.4.12), we have

$$x_{EV} = \Phi_{EV} u, \tag{3.1.2}$$

where Φ_{EV} denotes the utility elasticity of income, evaluated at current utility and initial prices. So

$$y_{EV} = n + \Phi_{EV} u, \tag{3.1.3}$$

and

$$d\,EV = \frac{1}{100} Y_{EV} (n + \Phi_{EV} u).$$

Although this equation is suitable for implementation, it does not lead directly to the GTAP 4.1 equation (3.1.1). To derive that we need to replace Y_{EV} with an expression involving $\bar{Y}$. Now

$$Y_{EV} = N X_{EV} = N_R \bar{N} X_{EV},$$

where X_{EV} denotes *per capita* expenditure required to achieve current utility at initial prices $\bar{\mathbf{P}}$; and

$$X_{EV} = U_R^{\Phi_{ARC}} \bar{X},$$

where Φ_{ARC} denotes the arc elasticity of income with respect to utility along the arc between $(\bar{\mathbf{P}}, \bar{U})$ and $(\bar{\mathbf{P}}, U)$; so

$$\begin{aligned} Y_{EV} &= N_R U_R^{\Phi_{ARC}} \bar{N} \bar{X} \\ &= N_R U_R^{\Phi_{ARC}} \bar{Y}, \end{aligned}$$

and

$$d\,EV = \frac{1}{100} N_R U_R^{\Phi_{ARC}} \bar{Y} (n + \Phi_{EV} u).$$

Suppose that initially Φ is equal to one. Then Φ_{EV} also is initially equal to one, since Φ_{EV} is initially equal to Φ. So, by continuity, Φ_{EV} is arbitrarily close to one for sufficiently small changes in U. Also, by the mean value theorem, Φ_{ARC} is arbitrarily close to the initial value of Φ, one, for sufficiently small changes in U. So, to a first-order approximation,

$$d\,EV = \frac{1}{100} N_R U_R \bar{Y} (n + u),$$

as was to be shown.■

3.2 Defects in the old treatment

As shown above (subsection 3.1), the old computation of EV is not exact, but is a valid approximation when the utility elasticity of income Φ is equal to one. Recalling proposition 4, we note that the condition is satisfied in standard GTAP data bases. Much like the old utility equation then (subsection 2.5), the EV equation is exactly accurate in Johansen simulations with data bases in which $\Phi = 1$ (including standard GTAP data bases); accurate to first order in multi-step simulations with data bases in which $\Phi = 1$; and inaccurate otherwise.

While the old treatment works well for standard GTAP data bases and small utility changes, a treatment that works well with non-standard data bases and large utility changes would of course be even better. This we now develop (subsection 3.3).

3.3 A new treatment

We seek a new formula for the equivalent variation that does not assume a unit elasticity of income with respect to utility, and is consistent with the new implementation of the regional household demand system.

We cannot implement equation (3.0.1) for EV directly, since we do not have an explicit functional form for the regional household expenditure function. Indeed, we do not have an explicit functional form even for the private consumption expenditure function. We can however compute the expenditure function indirectly, by implementing the demand system and solving for expenditure Y given utility U. It is then easy to compute EV.

The regional demand system already present in the model gives the relation between expenditure Y, current utility U, and current prices $\mathbf{P}$. To find the expenditure Y_{EV} required to achieve current utility U at initial prices $\bar{\mathbf{P}}$, we implement a *shadow* demand system with the same utility level as the ordinary system, but with prices held at initial levels. The expenditure level in this shadow system is just the Y_{EV} required to calculate EV.

Recalling the equation (3.1.2) for percentage change in equivalent income, $x_{EV} = \Phi_{EV} u$, we see that we can compute equivalent income provided that we track Φ_{EV}, the utility elasticity evaluated at current utility and initial prices. To track Φ_{EV} we need to compute the corresponding percentage change variable ϕ_{EV}. To do that we need to include in the shadow system most of the upper-level regional household demand system.

Furthermore, as shown by equation (2.13.4), $\phi = S_P \phi_P - b_{AV}$, the regional household elasticity ϕ depends on the private consumption elasticity

ϕ_P. To compute that elasticity, we need to include part of the private consumption demand system. The private consumption demand system also supplies to the top level system the change variable u_P for utility from private consumption required to update the levels coefficient U_P used in the top-level utility equation. Similarly the top-level demand system requires a variable u_G to be supplied from a government consumption demand system. Altogether then the shadow demand system includes four parts: a government consumption demand system, a private consumption demand system, a upper-level regional household demand system, and equations relating income to the equivalent variation.

3.4 Equivalent variation with preference change

So far we have not considered the effect of preference change on the equivalent variation. When the top level distribution parameters B_i or the scaling constant C change, should we calculate the equivalent variation at initial preferences or at final preferences, or should we include the effects of the preference change in the equivalent variation?

Extending our earlier notation, we may write $E(\mathbf{P}, U; \mathbf{A})$ for the generalized expenditure function evaluated at prices $\mathbf{P}$, utility U, and preferences $\mathbf{A}$. Initial income, $\bar{Y}$, is equal to $E(\bar{\mathbf{P}}, \bar{U}; \bar{\mathbf{A}})$, that is, to the expenditure function evaluated at initial prices, utility, and preferences. If we calculate the equivalent variation at initial preferences, then

$$EV = E(\bar{\mathbf{P}}, U; \bar{\mathbf{A}}) - E(\bar{\mathbf{P}}, \bar{U}; \bar{\mathbf{A}}); \tag{3.4.1}$$

if we calculate it at final preferences, then

$$EV = E(\bar{\mathbf{P}}, U; \mathbf{A}) - E(\bar{\mathbf{P}}, \bar{U}; \mathbf{A}); \tag{3.4.2}$$

if we include preference change in the equivalent variation, then

$$EV = E(\bar{\mathbf{P}}, U; \mathbf{A}) - E(\bar{\mathbf{P}}, \bar{U}; \bar{\mathbf{A}}). \tag{3.4.3}$$

Standard theory offers no guidance here, since it naturally considers the equivalent variation only with constant preferences. Likewise the old GTAP treatment contains no preference change variables. It does however face a comparable issue, namely the relationship between the equivalent variation and change in population. There its choice is to include population change in the equivalent variation, so that if prices and utility remain constant but population changes, the equivalent variation is non-zero.

We take a similar approach here, defining the equivalent variation to include change in preferences (3.4.3). Then we may write as before

$$EV = Y_{EV} - \bar{Y},$$

putting

$$Y_{EV} = E(\bar{\mathbf{P}}, U; \mathbf{A})$$

and

$$\bar{Y} = E(\bar{\mathbf{P}}, \bar{U}; \bar{\mathbf{A}}).$$

We now revert to our earlier concern (subsection 2.14) to neutralize the welfare effects of changes in the distribution parameters B_i using endogenous changes in the utility scaling factor C. Decomposing equation (3.4.3), we can isolate the effects of preference changes in either of two natural ways:

$$\begin{aligned}EV &= E(\bar{\mathbf{P}}, U; \mathbf{A}) - E(\bar{\mathbf{P}}, \bar{U}; \bar{\mathbf{A}}) \\ &= \Big(E(\bar{\mathbf{P}}, U; \mathbf{A}) - E(\bar{\mathbf{P}}, U; \bar{\mathbf{A}})\Big) + \Big(E(\bar{\mathbf{P}}, U; \bar{\mathbf{A}}) - E(\bar{\mathbf{P}}, \bar{U}; \bar{\mathbf{A}})\Big) \quad (3.4.4) \\ &= \Big(E(\bar{\mathbf{P}}, U; \mathbf{A}) - E(\bar{\mathbf{P}}, \bar{U}; \mathbf{A})\Big) + \Big(E(\bar{\mathbf{P}}, \bar{U}; \mathbf{A}) - E(\bar{\mathbf{P}}, \bar{U}; \bar{\mathbf{A}})\Big). \quad (3.4.5)\end{aligned}$$

Following the decomposition (3.4.4), we may let C vary so that

$$E(\bar{\mathbf{P}}, U; \mathbf{A}) - E(\bar{\mathbf{P}}, U; \bar{\mathbf{A}}) = 0; \tag{3.4.6}$$

then our specification (3.4.3) for EV reduces to specification (3.4.1), the equivalent variation calculated at initial preferences. Or following the decomposition (3.4.5), we may let

$$E(\bar{\mathbf{P}}, \bar{U}; \mathbf{A}) - E(\bar{\mathbf{P}}, \bar{U}; \bar{\mathbf{A}}) = 0; \tag{3.4.7}$$

in that case EV reduces to specification (3.4.2), the equivalent variation calculated at final preferences.

In implementing these conditions we already have available $E(\bar{\mathbf{P}}, \bar{U}; \bar{\mathbf{A}})$, since that is just $\bar{Y}$, and $E(\bar{\mathbf{P}}, U; \mathbf{A})$, since that is just Y_{EV}. So the only new values needed are $E(\bar{\mathbf{P}}, U; \bar{\mathbf{A}})$ and $E(\bar{\mathbf{P}}, \bar{U}; \mathbf{A})$. Like Y_{EV}, we can determine these in suitable shadow demand systems. For $E(\bar{\mathbf{P}}, U; \bar{\mathbf{A}})$, we hold fixed not only prices but also preferences; for $E(\bar{\mathbf{P}}, \bar{U}; \mathbf{A})$, we hold fixed prices and utility.

We have now three shadow demand systems to implement. Since they are all quite similar in form, it is convenient to implement them simultaneously, indexing the shadow variables and coefficients over a dummy parameter indicating the shadow system. As discussed above, (subsection 3.3), the module implementing these systems contains several parts: a government consumption demand system, a private consumption demand system, and a shadow top-level demand system. To these we now add a submodule to select coefficients and variables from the appropriate shadow system for EV calculations, and a submodule to determine distribution-parameter-neutralizing values for the scaling factor C.

3.5 Shared objects

We begin by declaring the coefficients and variables that the EV module contributes to the rest of the model. The module's primary function is to compute the regional and world-wide equivalent variations:

```
Variable (Change)(all,r,REG)
    EV(r) #equivalent variation, $ US million#;
Variable (Change)
    WEV #equivalent variation for the world#;
```

But it also contributes several coefficients needed for the EV decomposition (section 4): the utility elasticity of generalized expenditure Φ_{EV} in the EV shadow system:

```
Coefficient (all,r,REG)
    UTILELASEV(r)
    #utility elasticity of generalized expenditure, for EV calcs#;
```

and the quantities U_i of the goods in the shadow top-level demand system:

```
Coefficient (all,r,REG)
    UTILPRIVEV(r) #utility from private consumption, for EV calcs#;

Coefficient (all,r,REG)
    UTILGOVEV(r) #utility from private consumption, for EV calcs#;

Coefficient (all,r,REG)
    UTILSAVEEV(r) #utility from private consumption, for EV calcs#;
```

Next we define and characterize the set of shadow systems,

```
Set
    SHAD #shadow systems# (dp0u1, dp1u0, dp1u1);
```

Of these three systems, `dp0u1` uses initial preferences but current utility to calculate $E(\bar{\mathbf{P}}, U; \bar{\mathbf{A}})$; `dp1u0` uses current preferences but initial utility to calculate $E(\bar{\mathbf{P}}, \bar{U}; \mathbf{A})$; and `dp1u1` uses current preferences and utility to calculate $E(\bar{\mathbf{P}}, U; \mathbf{A}) = Y_{EV}$.

We characterize the shadow systems using two coefficients. The coefficient TRACKSDP is non-zero if the system tracks current preferences:

```
Coefficient (all,v,SHAD)
    TRACKSDP(v) #tracks preferences#;
Formula
    TRACKSDP("dp0u1") = 0;
Formula
    TRACKSDP("dp1u0") = 1;
Formula
    TRACKSDP("dp1u1") = 1;
```

while TRACKSU is non-zero if the system tracks current utility:

```
Coefficient (all,v,SHAD)
    TRACKSU(v) #tracks utility#;
Formula
    TRACKSU("dp0u1") = 1;
Formula
    TRACKSU("dp1u0") = 0;
Formula
    TRACKSU("dp1u1") = 1;
```

Finally we declare several variables shared between different parts of the EV module. The government consumption shadow demand system computes shadow utility from government consumption, ugshd, for use in the upper-level shadow demand system. The private consumption shadow demand system computes utility from private consumption, upshd, and the shadow elasticity of private consumption expenditure with respect to utility from private consumption, ueprivshd, both for use in the upper-level shadow demand system. The upper-level shadow demand system computes shadow government consumption expenditure, ygshd, for use in the government consumption demand system, and shadow private consumption expenditure, ypshd, for use in the private consumption demand system.

```
Variable (all,r,REG)(all,v,SHAD)
    ugshd(r,v) #per capita utility from gov't expend., shadow#;
Variable (all,r,REG)(all,v,SHAD)
    upshd(r,v) #per capita utility from private expend., shadow#;
Variable (all,r,REG)(all,v,SHAD)
    ueprivshd(r,v)
    #utility elasticity of private consn expenditure, shadow#;
Variable (all,r,REG)(all,v,SHAD)
    ygshd(r,v)
    #government consumption expenditure, in region r, shadow#;
```

```
Variable (all,r,REG)(all,v,SHAD)
    ypshd(r,v)
    #private consumption expenditure, in region r, shadow#;
```

From several coefficients and variables in the shadow top-level demand system, we need to extract the appropriate components for use in equivalent variation calculations. Thus the utility elasticity Φ_{EV} from the EV shadow system is extracted from the corresponding shadow-system-generic coefficient:

```
Coefficient (all,r,REG)(all,v,SHAD)
    UTILELASSHD(r,v)
    #elasticity of cost of utility wrt utility, shadow#;
```

Likewise for quantities U_i of goods in the top-level utility function:

```
Coefficient (all,r,REG)(all,v,SHAD)
    UTILPRIVSHD(r,v) #utility from private consumption#;

Coefficient (all,r,REG)(all,v,SHAD)
    UTILGOVSHD(r,v) #utility from government consumption#;

Coefficient (all,r,REG)(all,v,SHAD)
    UTILSAVESHD(r,v) #utility from saving#;
```

Regional income y has both shadow-system-generic and EV-system-specific variables, the latter used in the equivalent variation calculation:

```
Variable (all,r,REG)(all,v,SHAD)
    yshd(r,v) #regional household income, in region r, shadow#;

Variable (all,r,REG)
    yev(r) #equivalent income, for EV#;
```

and likewise for the corresponding income coefficients Y:

```
Coefficient (all,r,REG)(all,v,SHAD)
    INCOMESHD(r,v) #regional income, shadow#;

Coefficient (all,r,REG)
    INCOMEEV(r) #equivalent income, for EV#;
```

3.6 The shadow government consumption demand system

The task of the shadow government consumption demand system is to compute shadow values for the change variable u_G for utility from government consumption. It contains just one equation, a simplified version of the equation from the actual government consumption module (subsection 2.10) relating utility from government consumption to government consumption expenditure, with the price variable omitted:

```
Equation GOVUSHD
# utility from government consumption in r #
(all,r,REG)(all,v,SHAD)
    ygshd(r,v) - pop(r) = ugshd(r,v);
```

3.7 The shadow private consumption demand system

The task of the shadow private consumption demand system is to compute shadow values for the change variables u_P for utility from private consumption, and ϕ_P for the elasticity of private consumption expenditure with respect to utility from private consumption.

Recalling equation (2.4.10), we have, with fixed prices,

$$\phi_P = \sum_i S_{Ei}(u_{Pi} - x_P), \qquad (3.7.1)$$

where S_{Ei} denotes the expansion-parameter-weighted budget share, $S_i E_i/\Phi_P$, of commodity i in the shadow private consumption demand system. So to compute the shadow elasticity, we need shadow system values for the consumption shares S_{Ei} and the private consumption demands u_{Pi}. To compute the private consumption demands we need the expenditure elasticities, and to compute them and the expansion-parameter-weighted budget shares, we need the ordinary budget shares. To compute the ordinary budget shares, we need to record shadow private consumption expenditures for individual composite commodities.

We implement as a shadow system as much of the private consumption demand system as we need to compute the shadow private consumption budget shares. Since the shadow system uses the same theory as the ordinary private consumption demand system, we do not provide derivations, but instead refer the reader to the original documentation (Hertel and Tsigas [5]).

We begin by declaring the shadow private consumption demand variable:

```
Variable (all,i,TRAD_COMM)(all,r,REG)(all,v,SHAD)
    qpshd(i,r,v)
    #private hhld demand for commodity i in region r, shadow#;
```

We then define the shadow private consumption expenditure levels:

```
Coefficient (all,i,TRAD_COMM)(all,r,REG)(all,v,SHAD)
    VPASHD(i,r,v)
    #private hhld expend. on i in r valued at agent's prices, shadow#;
Formula (initial) (all,i,TRAD_COMM)(all,r,REG)(all,v,SHAD)
    VPASHD(i,r,v) = VPA(i,r);
Update (all,i,TRAD_COMM)(all,r,REG)(all,v,SHAD)
    VPASHD(i,r,v) = qpshd(i,r,v);
```

and the shadow private consumption budget shares:

```
Coefficient (all,r,REG)(all,v,SHAD)
    VPAREGSHD(r,v) #private consumption expenditure in region r, shadow#;
Formula (all,r,REG)(all,v,SHAD)
    VPAREGSHD(r,v) = sum{i,TRAD_COMM, VPASHD(i,r,v)};

Coefficient (all,i,TRAD_COMM)(all,r,REG)(all,v,SHAD)
    CONSHRSHD(i,r,v)
    #share of private hhld consn devoted to good i in r, shadow#;
Formula (all,i,TRAD_COMM)(all,r,REG)(all,v,SHAD)
    CONSHRSHD(i,r,v) = VPASHD(i,r,v)/VPAREGSHD(r,v);
```

We compute the expenditure elasticities as in the ordinary demand system, but using the shadow budget shares `CONSHRSHD` instead of the ordinary shares `CONSHR`:

```
Coefficient (all,i,TRAD_COMM)(all,r,REG)(all,v,SHAD)
    EYSHD(i,r,v)
    #expend. elast. of private hhld demand for i in r, shadow#;
Formula (all,i,TRAD_COMM)(all,r,REG)(all,v,SHAD)
    EYSHD(i,r,v)
  =      [1.0/[sum{n, TRAD_COMM, CONSHRSHD(n,r,v)*INCPAR(n,r)}]]
      * [
            INCPAR(i,r)*[1.0 - ALPHA(i,r)]
          + sum{n, TRAD_COMM, CONSHRSHD(n,r,v)*INCPAR(n,r)*ALPHA(n,r)}
        ]
    + ALPHA(i,r)
    - sum{n, TRAD_COMM, CONSHRSHD(n,r,v) * ALPHA(n,r)}
    ;
```

We can now compute the shadow private consumption demands, needed as shown above to update the levels coefficients for private consumption expenditure:

```
Equation PRIVDMNDSSHD
    #private hhld demands for composite commodities, shadow#
    (all,i,TRAD_COMM)(all,r,REG)(all,v,SHAD)
    qpshd(i,r,v) - pop(r) = EYSHD(i,r,v)*[ypshd(r,v) - pop(r)];
```

Next we compute the utility elasticity Φ_P of private consumption expenditure:

```
Coefficient (all,r,REG)(all,v,SHAD)
    UELASPRIVSHD(r,v)
    #elast. of cost wrt utility from private consn, shadow#;
Formula (all,r,REG)(all,v,SHAD)
    UELASPRIVSHD(r,v) = sum{i,TRAD_COMM, CONSHRSHD(i,r,v)*INCPAR(i,r)};
```

This appears both in the equation for utility u_P from private consumption, a simplified version of equation (2.11.1):

```
Equation PRIVATEUSHD
# computation of utility from private consumption in r (HT 45) #
(all,r,REG)(all,v,SHAD)
    ypshd(r,v) - pop(r) = UELASPRIVSHD(r,v)*upshd(r,v);
```

and as the denominator in the formula for the expansion-parameter-weighted budget shares S_{Ei}:

```
Coefficient (all,i,TRAD_COMM)(all,r,REG)(all,v,SHAD)
    XWCONSHRSHD(i,r,v)
    #expansion-parameter-weighted consumption share, shadow#;
Formula (all,i,TRAD_COMM)(all,r,REG)(all,v,SHAD)
    XWCONSHRSHD(i,r,v) = CONSHRSHD(i,r,v)*INCPAR(i,r)/UELASPRIVSHD(r,v);
```

With these shares, as shown in equation (3.7.1), we compute the change variable ϕ_P for the utility elasticity of private consumption expenditure:

```
Equation UTILELASPRIVSHD
    #elasticity of cost wrt utility from private consn, shadow#
    (all,r,REG)(all,v,SHAD)
    ueprivshd(r,v)
  = sum{i,TRAD_COMM, XWCONSHRSHD(i,r,v)*[qpshd(i,r,v) - ypshd(r,v)]};
```

3.8 The shadow upper-level regional household demand system

The tasks of the shadow upper-level regional household demand system are to compute shadow income and shadow private consumption expenditure, given utility. To compute shadow income, it tracks the elasticity Φ_{EV} of shadow income with respect to utility. Recalling equations (2.13.4), $\phi = S_P\phi_P - b_{AV}$, and (2.13.5), $b_{AV} = \sum_i S_i b_i$, we see that we must also compute shadow values for the upper level income disposition shares S_i, $i = P, G, S$. To do that we need shadow values for the upper-level components of income disposition, Y_P, Y_G, and Y_S; and to do that, we need shadow results for the related percentage change variables y_P, y_G, and q_S.

We begin by declaring a change variable for change in real saving:

```
Variable (all,r,REG)(all,v,SHAD)
    qsaveshd(r,v) #regional demand for NET savings, shadow#;
```

We then compute the level of income in the shadow system:

```
Formula (initial) (all,r,REG)(all,v,SHAD)
    INCOMESHD(r,v) = INCOME(r);
Update (all,r,REG)(all,v,SHAD)
    INCOMESHD(r,v) = yshd(r,v);
```

levels for the upper-level components of income disposition:

```
Coefficient (all,r,REG)(all,v,SHAD)
    PRIVEXPSHD(r,v)
    #private consumption expenditure in region r, shadow#;
Formula (initial) (all,r,REG)(all,v,SHAD)
    PRIVEXPSHD(r,v) = PRIVEXP(r);
Update (all,r,REG)(all,v,SHAD)
    PRIVEXPSHD(r,v) = ypshd(r,v);
!< PRIVEXPSHD should agree with VPAREGSHD.>!

Coefficient (all,r,REG)(all,v,SHAD)
    GOVEXPSHD(r,v)
    #government consumption expenditure in region r, shadow#;
Formula (initial) (all,r,REG)(all,v,SHAD)
    GOVEXPSHD(r,v) = GOVEXP(r);
Update (all,r,REG)(all,v,SHAD)
    GOVEXPSHD(r,v) = ygshd(r,v);

Coefficient (all,r,REG)(all,v,SHAD)
    SAVESHD(r,v)
    #saving in region r, shadow#;
Formula (initial) (all,r,REG)(all,v,SHAD)
    SAVESHD(r,v) = SAVE(r);
Update (all,r,REG)(all,v,SHAD)
    SAVESHD(r,v) = qsaveshd(r,v);
```

and upper-level income-disposition shares:

```
Coefficient (all,r,REG)(all,v,SHAD)
    XSHRPRIVSHD(r,v)
    #private expenditure share in regional income, shadow#;
```

```
Formula (all,r,REG)(all,v,SHAD)
    XSHRPRIVSHD(r,v) = PRIVEXPSHD(r,v)/INCOMESHD(r,v);

Coefficient (all,r,REG)(all,v,SHAD)
    XSHRGOVSHD(r,v)
    #government expenditure share in regional income, shadow#;
Formula (all,r,REG)(all,v,SHAD)
    XSHRGOVSHD(r,v) = GOVEXPSHD(r,v)/INCOMESHD(r,v);

Coefficient (all,r,REG)(all,v,SHAD)
    XSHRSAVESHD(r,v) #saving share in regional income, shadow#;
Formula (all,r,REG)(all,v,SHAD)
    XSHRSAVESHD(r,v) = SAVESHD(r,v)/INCOMESHD(r,v);
```

This enables us to compute the weighted average of the distribution parameters, following equation (2.13.5):

```
Variable (all,r,REG)(all,v,SHAD)
    dpavshd(r,v) #average distribution parameter shift, shadow#;
Equation DPARAVSHD #average distribution parameter shift, shadow#
    (all,r,REG)(all,v,SHAD)
    dpavshd(r,v)
  =   XSHRPRIVSHD(r,v)*dppriv(r)
    + XSHRGOVSHD(r,v)*dpgov(r)
    + XSHRSAVESHD(r,v)*dpsave(r)
    ;
```

and the utility elasticity of income, following equation (2.13.4), but using dummy coefficients to control whether the distributional coefficients affect the result in each shadow system:

```
Variable (all,r,REG)(all,v,SHAD)
    uelasshd(r,v)
    #elasticity of cost of utility wrt utility, shadow#;
Equation UTILITELASTICSHD
    #elasticity of cost of utility wrt utility, shadow#
    (all,r,REG)(all,v,SHAD)
    uelasshd(r,v) = XSHRPRIV(r)*ueprivshd(r,v) - TRACKSDP(v)*dpavshd(r,v);
```

This in turn enables us to implement the upper-level demand equations, following equations (2.13.1)–(2.13.3), again using dummy coefficients on the distribution parameters:

```
Equation PCONSEXPSHD #private consumption expenditure, shadow#
    (all,r,REG)(all,v,SHAD)
    ypshd(r,v) - yshd(r,v)
```

```
  = -[ueprivshd(r,v) - uelasshd(r,v)] + TRACKSDP(v)*dppriv(r);

Equation GOVCONSEXPSHD #government consumption expenditure#
    (all,r,REG)(all,v,SHAD)
    ygshd(r,v) - yshd(r,v) = uelasshd(r,v) + TRACKSDP(v)*dpgov(r);

Equation SAVINGSHD #saving# (all,r,REG)(all,v,SHAD)
    qsaveshd(r,v) - yshd(r,v) = uelasshd(r,v) + TRACKSDP(v)*dpsave(r);
```

and to compute the level of the utility elasticity of income:

```
Formula (initial) (all,r,REG)(all,v,SHAD)
    UTILELASSHD(r,v) = UTILELAS(r);
Update (all,r,REG)(all,v,SHAD)
    UTILELASSHD(r,v) = uelasshd(r,v);
```

We also define levels coefficients for the goods in the top-level utility function:

```
Formula (initial) (all,r,REG)(all,v,SHAD)
    UTILPRIVSHD(r,v) = UTILPRIV(r);
Update (all,r,REG)(all,v,SHAD)
    UTILPRIVSHD(r,v) = upshd(r,v);

Formula (initial) (all,r,REG)(all,v,SHAD)
    UTILGOVSHD(r,v) = UTILGOV(r);
Update (all,r,REG)(all,v,SHAD)
    UTILGOVSHD(r,v) = ugshd(r,v);

Formula (initial) (all,r,REG)(all,v,SHAD)
    UTILSAVESHD(r,v) = UTILSAVE(r);
Update (change) (all,r,REG)(all,v,SHAD)
    UTILSAVESHD(r,v) = 100*[qsaveshd(r,v) - pop(r)]*UTILSAVESHD(r,v);
```

Finally we compute the percentage change in shadow income, following equation (2.14.1), and using dummy coefficients on utility, the utility scaling factor, and the distribution parameters:

```
Equation INCOME_SHAD #shadow income# (all,r,REG)(all,v,SHAD)
  TRACKSU(v)*u(r) = TRACKSDP(v)*au(r)
    + TRACKSDP(v)*DPARPRIV(r)*loge(UTILPRIVSHD(r,v))*dppriv(r)
    + TRACKSDP(v)*DPARGOV(r)*loge(UTILGOVSHD(r,v))*dpgov(r)
    + TRACKSDP(v)*DPARSAVE(r)*loge(UTILSAVESHD(r,v))*dpsave(r)
    + [1.0/UTILELASSHD(r,v)]*[yshd(r,v) - pop(r)];
```

3.9 Extracting coefficients and variables for the EV calculations

The next part of the module is a piece of bridging code selecting coefficients and variables from the appropriate shadow system for use in equivalent variation calculations. For use in the EV decomposition (section 4) we define the utility elasticity Φ_{EV} from the EV shadow system:

```
Formula (all,r,REG)
    UTILELASEV(r) = UTILELASSHD(r,"dp1u1");
```

and the quantities U_i of the goods in the shadow top-level demand system:

```
Formula (all,r,REG)
    UTILPRIVEV(r) = UTILPRIVSHD(r,"dp1u1");

Formula (all,r,REG)
    UTILGOVEV(r) = UTILGOVSHD(r,"dp1u1");

Formula (all,r,REG)
    UTILSAVEEV(r) = UTILSAVESHD(r,"dp1u1");
```

For use in calculating the equivalent variation itself we define the levels coefficient Y_{EV} and change variable y_{EV} for income in the EV shadow system:

```
Formula (all,r,REG)
    INCOMEEV(r) = INCOMESHD(r,"dp1u1");

Equation INCOME_EQUIV #equivalent income# (all,r,REG)
    yev(r) = yshd(r,"dp1u1");
```

3.10 Computing the equivalent variation

The income variable and coefficient are for local use, the utility elasticity for the welfare decomposition (subsection 4.3).

Implementing equation (3.0.1), we compute regional equivalent variation EV:

```
Equation EVREG #regional EV (HT 67)# (all,r,REG)
    EV(r) = [INCOMEEV(r)/100.0]*yev(r);
```

We also compute a world equivalent variation, `WEV`, as the sum of the regional equivalent variations:

```
Equation EVWLD #EV for the world (HT 68)#
    WEV = sum{r, REG, EV(r)};
```

3.11 Neutralizing the welfare effects of distributional parameter change

We provide an equation and slack variable to control the neutralization of distributional parameter changes at the initial utility level, following equation (3.4.7):

```
Variable (all,r,REG)
    yu0slack(r) #controls normalization of preferences at original U#;
Equation NORMPREFU0 #normalize preferences at original U# (all,r,REG)
    yshd(r,"dp0u1") = yu0slack(r);
```

When the variable `yu0slack` is exogenous and zero, equation (3.4.7) applies, and the equivalent variation reduces to the specification (3.4.2). Similarly, we provide an equation and slack variable for neutralization at the final utility level, following equation (3.4.6):

```
Variable (all,r,REG)
    yu1slack(r) #controls normalization of preferences at final U#;
Equation NORMPREFU1 #normalize preferences at final U# (all,r,REG)
    yshd(r,"dp1u1") = yshd(r,"dp1u0") + yu1slack(r);
```

When the variable `yu1slack` is exogenous and zero, equation (3.4.6) applies, and the equivalent variation reduces to the specification (3.4.1).

4 Decomposing the equivalent variation

We describe the old decomposition of the equivalent variation (subsection 4.1), discuss its defects (subsection 4.2), and derive (subsection 4.3, 4.4) and implement (subsection 4.5) a new decomposition.

In the derivations below, we divide each EV decomposition formula into two parts: a lengthy formula decomposing some income-related variable, such as real income or real *per capita* income, and a *decomposition scheme* relating the income variable to EV. Substituting the decomposition of the income-related variable into the decomposition scheme yields the full EV decomposition.

4.1 The old treatment

The old derivation (Huff and Hertel [6]) uses a decomposition of real income,

$$D = Y(y - p), \tag{4.1.1}$$

where D stands for a rather lengthy decomposition (reproduced with minor changes in subsection 4.3) of real regional income into components related to factor endowments, technological change, allocative efficiency, and terms of trade. The relation between real income and EV is given by the decomposition scheme

$$d\,EV = \frac{1}{100} U_R N_R \frac{\bar{Y}}{Y} \left[D - \left(\sum_i Y_{Pi}(E_i - 1) \right) u_P \right], \tag{4.1.2}$$

where Y_{Pi} denotes private consumption expenditure on commodity i.

The problems with the old decomposition relate not to the real income decomposition but to the decomposition scheme. Accordingly, we do not derive here the real income decomposition, but refer the reader to the original documentation. We do provide a new derivation for the decomposition scheme, in order to identify sources of error in the old decomposition, and also to explain why the old decomposition is consistent with the old computation of EV.

We use the old utility equation (2.5.1),

$$\hat{u} = S_P u_P + S_G(q_G - n) + S_S(q_S - n). \tag{4.1.3}$$

We recall (from subsection 2.5) that the old computation is invalid in general, but valid in the special case $\Phi = 1,\ \Phi_P = 1$, and that standard GTAP data bases fall within the special case. We use for this derivation the notation $\hat{u}$ for utility computed according to equation (4.1.3).

Recalling equations (2.1.10) and (2.1.11), and dropping the government consumption and saving slack variables κ_G and κ_S, we have

$$q_G = y - p_G,$$
$$q_S = y - p_S.$$

Also, from equation (2.11.1), we have $\Phi_P u_P = y_P - n - p_P$. Adding $u_P - \Phi_P u_P$ to both sides, and putting y for y_P (consistent with the old treatment provided the slack variables are zero), we obtain

$$u_P = y - n - p_P - (\Phi_P - 1)u_P.$$

Substituting into equation (4.1.3), we obtain

$$\hat{u} = y - n - (S_P p_P + S_G p_G + S_S p_S) - S_P(\Phi_P - 1)u_P.$$

Then substituting from equation (2.4.13), $p = S_P p_P + S_G p_G + S_S p_S$, we obtain

$$\hat{u} = y - n - p - S_P(\Phi_P - 1)u_P.$$

Substituting into the old EV equation (3.1.1), $d\,EV = (1/100)U_R N_R \bar{Y}(n + \hat{u})$, we obtain

$$d\,EV = \frac{1}{100}U_R N_R \bar{Y}[y - p - S_P(\Phi_P - 1)u_P].$$

Substituting for D from equation (4.1.1), we obtain

$$d\,EV = \frac{1}{100}U_R N_R \frac{\bar{Y}}{Y}[D - Y_P(\Phi_P - 1)u_P].$$

Substituting for Φ_P from equation (2.4.8), $\Phi_P = \sum_i S_i^P E_i$, we obtain finally the old EV decomposition scheme,

$$d\,EV = \frac{1}{100}U_R N_R \frac{\bar{Y}}{Y}\left[D - \left(\sum_i Y_{Pi}(E_i - 1)\right)u_P\right].$$

4.2 Defects in the old treatment

The old welfare decomposition has two defects: it contains a nuisance term, the term in u_P in equation (4.1.2); and it is in general invalid.

As shown in subsection 3.1, the old decomposition relies on the old utility equation (4.1.3), and inherits its validity conditions. Accordingly, it is valid in Johansen simulations with data bases in which $\Phi = \Phi_P = 1$ (including standard GTAP data bases); approximate in non-linear simulations in which initially $\Phi = \Phi_P = 1$; and invalid otherwise.

While this is the major defect of the old decomposition, it is also in a way a merit, since it allows the decomposition to be consistent with the old EV computation. More specifically, the old EV computation and decomposition are consistent because they use the same equation (4.1.3) for aggregate utility.

4.3 A revised treatment

Hanslow [4] presents a general welfare decomposition applicable to many CGE models. For convenience, we base our derivation on the GTAP-specific Huff and Hertel ([6]) approach. As revised, the results are consistent with the Hanslow decomposition.

In revising the decomposition, we at first assume no changes in preferences, and then extend our results to incorporate preference changes. Rearranging equation (2.4.12), $x = p + \Phi u$, we obtain

$$\begin{aligned} u &= \Phi^{-1}(x - p) \\ &= \Phi^{-1}(y - p - n) && \text{by defn. of } x \\ &= \Phi^{-1}(Y^{-1}D - n) && \text{from (4.1.1)} \end{aligned}$$

Substituting into equation (3.1.3), $y_{EV} = \Phi_{EV}u + n$, we obtain

$$y_{EV} = \frac{\Phi_{EV}}{\Phi}(Y^{-1}D - n) + n.$$

Then substituting into equation (3.0.1), $d\,EV = \frac{1}{100}Y_{EV}y_{EV}$, we obtain the decomposition scheme

$$d\,EV = \frac{1}{100}\frac{\Phi_{EV}}{\Phi}\frac{Y_{EV}}{Y}D - \frac{1}{100}\left[\frac{\Phi_{EV}}{\Phi} - 1\right]Y_{EV}n.$$

This scheme suffers from one objectionable feature, the presence of a nuisance term involving population growth n. In simulations with standard data bases (with $\Phi_{EV} = \Phi = 1$ initially), the term would typically be small but non-zero. We can remove this nuisance by modifying the income decomposition, to decompose not real regional income $y - p$ but real *per capita* income $x - p$. Accordingly we write

$$Y(x - p) = D^*, \tag{4.3.1}$$

where D^* represents a decomposition of real *per capita* income. Then proceeding as before, we obtain

$$u = \Phi^{-1}Y^{-1}D^*,$$

$$y_{EV} = \frac{\Phi_{EV}}{\Phi}Y^{-1}D^* + n,$$

and

$$d\,EV = \frac{1}{100}\frac{\Phi_{EV}}{\Phi}\frac{Y_{EV}}{Y}D^* + \frac{1}{100}Y_{EV}n.$$

Now instead of a nuisance term, we have an interpretable term in population growth n.

Finally, we incorporate preference changes. Instead of the simpler equation (2.4.12), we begin with the more complete equation (2.14.1),

$$
\begin{aligned}
u &= c + \sum_i B_i(\log U_i) b_i + \Phi^{-1}(x - p) \\
&= c + \sum_i B_i(\log U_i) b_i + \Phi^{-1} Y^{-1} D^*,
\end{aligned}
$$

substituting from equation (4.3.1). Also, adapting equation (2.14.1), we have

$$
y_{EV} = \Phi_{EV}\Big(-c - \sum_i B_i(\log U_{EVi}) b_i + u\Big) + n,
$$

where U_{EVi} denotes the level of good i in the top-level utility function, in the shadow demand system with initial prices but current utility and preferences. Then proceeding as before, we obtain

$$
\begin{aligned}
d\,EV = &-\frac{1}{100}\Phi_{EV} Y_{EV} \sum_i B_i \left(\log \frac{U_{EVi}}{U_i}\right) b_i \\
&+ \frac{1}{100}\frac{\Phi_{EV}}{\Phi}\frac{Y_{EV}}{Y} D^* + \frac{1}{100} Y_{EV} n.
\end{aligned}
\tag{4.3.2}
$$

We note that changes c in the utility scaling factor do not affect the equivalent variation, and that changes in the distribution parameters affect it only when correlated with differences between the actual and shadow subutilities U_i and U_{EVi}. If both distribution parameter changes and price changes favor usage of top-level good i, then the effect of the distribution parameter changes on utility is more favorable with final prices than with initial prices, so expenditure in the shadow system needs to be higher than it would otherwise, so the contribution to the equivalent variation is positive. Conversely, if the distribution parameter for good i increases while price changes operate to discourage its consumption, the contribution to the equivalent variation is negative.

4.4 Decomposing real *per capita* income

Based on Huff and Hertel [6], we have a decomposition of real regional income:

```
(all,r,REG)
    INCOME(r)*[y(r) - p(r)]
=   sum{i, ENDW_COMM, VOA(i,r)*qo(i,r)} - VDEP(r)*kb(r)
  + sum{j, PROD_COMM, VOA(j,r)*ao(j,r)}
```

```
  + sum{j, PROD_COMM, VVA(j,r)*ava(j,r)}
  + sum{j, PROD_COMM, sum{i, ENDW_COMM, VFA(i,j,r)*afe(i,j,r)}}
  + sum{j, PROD_COMM, sum{i, TRAD_COMM, VFA(i,j,r)*af(i,j,r)}}
  + sum{s, REG, sum{i, TRAD_COMM, sum{m, MARG_COMM,
      VTMFSD(m,i,s,r)*atmfsd(m,i,r,s)
    }}}
  + sum{i, NSAV_COMM, [VOM(i,r) - VOA(i,r)]*qo(i,r)}
  + sum{i, ENDW_COMM, sum{j, PROD_COMM,
        [VFA(i,j,r) - VFM(i,j,r)]*qfe(i,j,r)
    }}
  + sum{j, PROD_COMM, sum{i, TRAD_COMM,
        [VIFA(i,j,r) - VIFM(i,j,r)]*qfm(i,j,r)
    }}
  + sum{j, PROD_COMM, sum{i, TRAD_COMM,
        [VDFA(i,j,r) - VDFM(i,j,r)]*qfd(i,j,r)
    }}
  + sum{i, TRAD_COMM, [VIPA(i,r) - VIPM(i,r)]*qpm(i,r)}
  + sum{i, TRAD_COMM, [VDPA(i,r) - VDPM(i,r)]*qpd(i,r)}
  + sum{i, TRAD_COMM, [VIGA(i,r) - VIGM(i,r)]*qgm(i,r)}
  + sum{i, TRAD_COMM, [VDGA(i,r) - VDGM(i,r)]*qgd(i,r)}
  + sum{i, TRAD_COMM, sum{s, REG,
        [VXWD(i,r,s) - VXMD(i,r,s)]*qxs(i,r,s)
    }}
  + sum{i, TRAD_COMM, sum{s, REG,
        [VIMS(i,s,r) - VIWS(i,s,r)]*qxs(i,s,r)
    }}
  + sum{i, TRAD_COMM, sum{s, REG, VXWD(i,r,s)*pfob(i,r,s)}}
  + sum{m, MARG_COMM, VST(m,r)*pm(m,r)}
  - sum{i, TRAD_COMM, sum{s, REG, VXWD(i,s,r)*pfob(i,s,r)}}
  - sum{m, MARG_COMM, VTMD(m,r)*pt(m)}
  + NETINV(r)*pcgds(r) - SAVE(r)*psave(r)
  ;
```

This is a equation from Huff and Hertel [6], modified to conform to the new (in GTAP 5) treatment of international margins. The right hand side is the expression represented above as D. Rearranging, substituting for the coefficient `INCOME` on `pop`, and introducing from GTAP 5 new notation for tax revenue coefficients, we obtain a decomposition for real *per capita* income:

```
(all,r,REG)
    INCOME(r)*[y(r) - pop(r) - p(r)]
=   sum{i,ENDW_COMM, VOA(i,r)*[qo(i,r) - pop(r)]}
  - VDEP(r)*[kb(r) - pop(r)]
  + sum{j,PROD_COMM, VOA(j,r)*ao(j,r)}
```

```
+ sum{j,PROD_COMM, VVA(j,r)*ava(j,r)}
+ sum{j,PROD_COMM, sum{i,ENDW_COMM, VFA(i,j,r)*afe(i,j,r)}}
+ sum{j,PROD_COMM, sum{i,TRAD_COMM, VFA(i,j,r)*af(i,j,r)}}
+ sum{s,REG, sum{i,TRAD_COMM, sum{m,MARG_COMM,
    VTMFSD(m,i,s,r)*atmfsd(m,i,r,s)
  }}}
+ sum{i,NSAV_COMM, PTAX(i,r)*[qo(i,r) - pop(r)]}
+ sum{i,ENDW_COMM, sum{j,PROD_COMM,
      ETAX(i,j,r)*[qfe(i,j,r) - pop(r)]
  }}
+ sum{j,PROD_COMM, sum{i,TRAD_COMM,
      IFTAX(i,j,r)*[qfm(i,j,r) - pop(r)]
  }}
+ sum{j,PROD_COMM, sum{i,TRAD_COMM,
      DFTAX(i,j,r)*[qfd(i,j,r) - pop(r)]
  }}
+ sum{i,TRAD_COMM, IPTAX(i,r)*[qpm(i,r) - pop(r)]}
+ sum{i,TRAD_COMM, DPTAX(i,r)*[qpd(i,r) - pop(r)]}
+ sum{i,TRAD_COMM, IGTAX(i,r)*[qgm(i,r) - pop(r)]}
+ sum{i,TRAD_COMM, DGTAX(i,r)*[qgd(i,r) - pop(r)]}
+ sum{i,TRAD_COMM, sum{s,REG,
      XTAXD(i,r,s)*[qxs(i,r,s) - pop(r)]
  }}
+ sum{i,TRAD_COMM, sum{s,REG,
      MTAX(i,s,r)*[qxs(i,s,r) - pop(r)]
  }}
+ sum{i,TRAD_COMM, sum{s,REG, VXWD(i,r,s)*pfob(i,r,s)}}
+ sum{m,MARG_COMM, VST(m,r)*pm(m,r)}
- sum{i,TRAD_COMM, sum{s,REG, VXWD(i,s,r)*pfob(i,s,r)}}
- sum{m,MARG_COMM, VTMD(m,r)*pt(m)}
+ NETINV(r)*pcgds(r)
- SAVE(r)*psave(r)
;
```

Here the right hand side is the expression referred to above as D^*.

Unlike for example Hanslow [4], we do not introduce into the decomposition a new term involving population. Instead we incorporate the population variable into the terms involving quantity variables. We prefer this approach for several reasons.

- Looking forward to the equivalent variation decomposition, it does not create there a nuisance term involving population growth. There is indeed still a population growth term. It is however no longer a nuisance term but an interpretable term, expressing the intuition that in the absence of imbalances in growth, income grows equiproportionally with population.

- It does lead to a redefinition of the endowment terms. We recognize now an increase in utility arising not from growth in total endowments, but from growth in endowments *per capita*. While change admittedly is bad, this change is not very bad, since the new endowment terms are as readily interpretable as the old ones.

- It leads also to a redefinition of the allocative efficiency effects, but here the change is for the better. With balanced growth in a distorted economy, the old decomposition reported an allocative efficiency improvement associated with every taxed flow, and an allocative efficiency deterioration associated with every subsidised flow. Intuitively however, balanced growth involves no change in allocative efficiency. The new decomposition here conforms to intuition better than the old.

4.5 Implementation

To implement the new treatment, we need to define the new population growth term in the decomposition, and revise the old terms. The old terms included a factor representing $\frac{U}{\bar{U}}\frac{N}{\bar{N}}\bar{Y}/Y$. The new terms include instead a factor representing $(\Phi_{EV}/\Phi)(Y_{EV}/Y)$. Since the numerator $\frac{U}{\bar{U}}\frac{N}{\bar{N}}\bar{Y}$ in the old factor is an approximation to Y_{EV} (provided that the elasticity of income with respect to utility is initially equal to one), and since Φ_{EV}/Φ in the new factor is (for small changes) approximately equal to one, the old factor may be considered an approximation to the new one.

To implement the new treatment, we first compute the equivalent variation scaling factor $(\Phi_{EV}/\Phi)(Y_{EV}/Y)$:

```
Coefficient (all,r,REG)
    EVSCALFACT(r) #equivalent variation scaling factor#;
Formula (all,r,REG)
    EVSCALFACT(r)
  = [UTILELASEV(r)/UTILELAS(r)]*[INCOMEEV(r)/INCOME(r)];
```

We then revise the decomposition-based computation of equivalent variation, using equation (4.3.2) and the real *per capita* income decomposition obtained in subsection 4.4.

```
Variable (Linear,Change)(all,r,REG)
    EV_ALT(r) # regional EV computed in alternative way #;
Equation EV_DECOMPOSITION
```

```
# decomposition of Equivalent Variation #
(all,r,REG)
    EV_ALT(r)
  =   0.01*UTILELASEV(r)*INCOMEEV(r)*[
          DPARPRIV(r)*loge(UTILPRIVEV(r)/UTILPRIV(r))*dppriv(r)
        + DPARGOV(r)*loge(UTILGOVEV(r)/UTILGOV(r))*dpgov(r)
        + DPARSAVE(r)*loge(UTILSAVEEV(r)/UTILSAVE(r))*dpsave(r)
      ]
    +   [0.01*EVSCALFACT(r)]
      * [
            sum{i,NSAV_COMM, PTAX(i,r)*[qo(i,r) - pop(r)]}
          + sum{i,ENDW_COMM, sum{j,PROD_COMM,
              ETAX(i,j,r)*[qfe(i,j,r) - pop(r)]
            }}
          + sum{j,PROD_COMM, sum{i,TRAD_COMM,
              IFTAX(i,j,r)*[qfm(i,j,r) - pop(r)]
            }}
          + sum{j,PROD_COMM, sum{i,TRAD_COMM,
              DFTAX(i,j,r)*[qfd(i,j,r) - pop(r)]
            }}
          + sum{i,TRAD_COMM, IPTAX(i,r)*[qpm(i,r) - pop(r)]}
          + sum{i,TRAD_COMM, DPTAX(i,r)*[qpd(i,r) - pop(r)]}
          + sum{i,TRAD_COMM, IGTAX(i,r)*[qgm(i,r) - pop(r)]}
          + sum{i,TRAD_COMM, DGTAX(i,r)*[qgd(i,r) - pop(r)]}
          + sum{i,TRAD_COMM, sum{s,REG,
              XTAXD(i,r,s)*[qxs(i,r,s) - pop(r)]
            }}
          + sum{i,TRAD_COMM, sum{s,REG,
              MTAX(i,s,r)*[qxs(i,s,r) - pop(r)]
            }}
          + sum{i,ENDW_COMM, VOA(i,r)*[qo(i,r) - pop(r)]}
          - VDEP(r)*[kb(r) - pop(r)]
          + sum{i,PROD_COMM, VOA(i,r)*ao(i,r)}
          + sum{j,PROD_COMM, VVA(j,r)*ava(j,r)}
          + sum{i,ENDW_COMM, sum{j,PROD_COMM, VFA(i,j,r)*afe(i,j,r)}}
          + sum{j,PROD_COMM, sum{i,TRAD_COMM, VFA(i,j,r)*af(i,j,r)}}
          + sum{m,MARG_COMM, sum{i,TRAD_COMM,
              sum{s,REG, VTMFSD(m,i,s,r)*atmfsd(m,i,s,r)}}}
          + sum{i,TRAD_COMM, sum{s,REG, VXWD(i,r,s)*pfob(i,r,s)}}
          + sum{m,MARG_COMM, VST(m,r)*pm(m,r)}
          + NETINV(r)*pcgds(r)
          - sum{i,TRAD_COMM, sum{s,REG, VXWD(i,s,r)*pfob(i,s,r)}}
          - sum{m,MARG_COMM, VTMD(m,r)*pt(m)}
          - SAVE(r)*psave(r)
        ]
```

```
    + 0.01*INCOMEEV(r)*pop(r);
```

Consistency between this and the standard equivalent variation computation is a check on the validity of the decomposition.

Finally we compute various components of the change in equivalent variation. We compute first the distributional parameter component:

```
Variable (Linear,Change) (all,r,REG) CNTdpar(r)
    # contribution to EV of change in distribution parameters#;
Equation CNT_WEV_dpar (all,r,REG)
    CNTdpar(r) = 0.01*UTILELASEV(r)*INCOMEEV(r)*[
          DPARPRIV(r)*loge(UTILPRIVEV(r)/UTILPRIV(r))*dppriv(r)
        + DPARGOV(r)*loge(UTILGOVEV(r)/UTILGOV(r))*dpgov(r)
        + DPARSAVE(r)*loge(UTILSAVEEV(r)/UTILSAVE(r))*dpsave(r)
      ];
```

and the population component:

```
Variable (Linear,Change) (all,r,REG) CNTpop(r)
    #contribution to EV in region r of change in population#;
Equation CONT_WEV_pop (all,r,REG)
    CNTpop(r) = 0.01*INCOMEEV(r)*pop(r);
```

The other components derive from the real *per capita* income decomposition. They are generally similar to the corresponding components of the old decomposition, but with the new scaling factor replacing the old. For instance, for the allocative efficiency effect associated with production subsidies and income taxes, we have corresponding to the first term in the decomposition:

```
Variable (Linear,Change) (all,r,REG) CNTqor(r)
    #contribution to EV in region r of output changes#;
Equation  CONT_WEV_qor (all,r,REG)
    CNTqor(r)
  = sum{i,NSAV_COMM, 0.01*EVSCALFACT(r)*PTAX(i,r)*[qo(i,r) - pop(r)]};
```

The code for the remaining components is not reproduced here but may be found in the associated program source file.

5 Properties of the final demand system

In the new treatment, the model should display several easily checked properties:

- All variables except utility from private consumption (ϕ_P or `uelaspriv`) are invariant with respect to rescalings of the CDE expansion parameters.

- All variables except aggregate utility (ϕ or `uelas`) are invariant with respect to changes in the initial level of the elasticity of income with respect to utility.
- In quantity homogeneity tests—that is, simulations in which uniform shocks are applied to population, factor endowments, and any other exogenous quantity variables—all components of the EV decomposition except the population component are zero.

The first property is not obvious, since rescalings of the expansion parameters do affect the utility elasticity of private consumption expenditure, and, in the linearised equation system, changes in the utility elasticity of private consumption expenditure affect the upper-level allocation of income. Nevertheless, as the following proposition shows, it does apply:

Proposition 9 *With a upper-level Cobb-Douglas demand system and a bottom-level CDE system, with distribution parameters calibrated to a given initial equilibrium, rescaling the CDE expansion parameters has no effect on quantities demanded.*

Proof. Suppose that the CDE expansion parameters E_i are rescaled by a common positive factor K. Now changing the maximand from U to U^K does not affect quantities demanded, but as we see by substituting into equation (2.1.1), maximizing U^K with the old expansion parameters is equivalent to maximizing U with the new expansion parameters. So rescaling the CDE expansion parameters does not affect the private consumption demand system. It does affect the upper level of the final demand system, since the elasticity of private consumption expenditure with respect to utility from private consumption is linearly homogeneous in the CDE expansion parameters. To calibrate to the observed income allocation, however, when we rescale the expansion parameters by a factor K, we need also to multiply by K the upper-level distribution factor for private consumption. With that adjustment, the new demand system is equivalent to the old.■

In the private consumption demand system, as income increases, the budget share of commodities with higher expansion parameters increases. Then because of the expansion-parameter weighting of `XWCONSHR`, the utility elasticity `uepriv` also increases. This leads to a shift away from private consumption toward government consumption and saving. In addition, reductions in relative prices of commodities with low expansion parameters (with sufficiently low price elasticities) typically decrease their budget share, again leading to increases in `uepriv` and reallocation of income away from private consumption toward government consumption and saving.

In standard GTAP data bases, the greatest differences in expansion parameters are typically between food and non-food commodities, the expansion parameters of food commodities typically being much lower than those of non-food. Accordingly, the share of private consumption in regional income typically varies directly with food prices.

Simulations with the old and new models with aggregated standard data bases suggest the following tentative generalizations:

- In moving from the old to the new treatment, corrections to the welfare variables are typically small.
- Under the new treatment, the upper-level allocation of income is typically insensitive to changes in income.
- Under the new treatment, the upper-level allocation of income is moderately sensitive to changes in relative prices of commodities with different expansion parameters. In particular, in low- and middle-income countries, the upper-level allocation of income is moderately sensitive to changes in the price of food relative to other commodities. The upper-level allocation is typically less sensitive to food prices in high income countries, since there the share of food in private consumption expenditure is typically low.

6 Future work

Later versions of this paper may include more detailed derivations and some illustrative applications. We may revise the program `decomp.tab`, used to prepare a ViewHAR-friendly equivalent variation decomposition report, to work with the revised `gtap.tab`. We may assess the empirical merits of the new, more complex upper-level demand system.

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