An Introduction to Systematic Sensitivity Analysis via Gaussian Quadrature

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Abstract

Economists recognize that results from simulation models are dependent, sometimes highly dependent, on values employed for critical exogenous variables. To account for this, analysts sometimes conduct sensitivity analysis with respect to key exogenous variables. This paper presents a practical approach for conducting systematic sensitivity analysis, called Gaussian quadrature. The approach views key exogenous variables as random variables with associated distributions. It produces estimates of means and standard deviations of model results while requiring a limited number of solves of the model. Under mild conditions, all of which hold with respect to the GTAP model, there is strong reason to believe that the estimates of means and standard deviations will be quite accurate.

Table of Contents

1 Introduction 1

2 Economic Simulation with Random Exogenous Variables: A Problem of Numerical Integration 2

3 Conclusion 11

References 12

Tables

Table 1 Absolute Value of Univariate GQ Approximation Error To Equation (11) 7

Table 2 Estimated Rate of Convergence to True Values for Results in Table 1 7

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1 Introduction

Economists recognize that results from simulation models are dependent, sometimes highly dependent, on values employed for critical exogenous variables. To account for this, analysts sometimes conduct sensitivity analysis with respect to key exogenous variables. Examples of key exogenous variables include behavioral parameters, endowments, and policy distortions such as taxes. However, as currently practiced, sensitivity analysis in large models is typically ad hoc. As an example of ad hoc sensitivity analysis, analysts might examine different results from a simulation model for a relatively small, arbitrarily determined set of values for key exogenous variables. While lending some insight into the robustness of model results with respect to variation in certain exogenous variables, the method is far from systematic in exploring the effects of different combinations of exogenous variables.

A simple and effective approach to systematic sensitivity analysis exists. Monte Carlo analysis, with a sufficient number of repetitions,[1] would yield reliable and systematic insights into the impacts of variation in key exogenous variables on model results. However, if the model is large, such as a global computable general equilibrium model, a Monte Carlo approach quickly becomes impractical. For example, if the model takes five minutes to solve (hardly out of the ordinary), 1,000 Monte Carlo repetitions would take nearly 3.5 days.

In this technical paper, a practical approach for conducting systematic sensitivity analysis in large models, called Gaussian Quadrature (GQ), is presented. The GQ approach views key exogenous variables as random variables with associated distributions. If exogenous variables are random, then model results are random. As outputs, the GQ approach produces estimates of means and standard deviations of model results. Under mild conditions, all of which hold with respect to the GTAP model, the estimates of means and standard deviations will be quite accurate. For an example of an application of this approach, see ARNDT and HERTEL.

The paper is structured as follows. First, we present economic simulations with random exogenous variables as problems of numerical integration. Second, we review some of the numerical integration

1. To obtain 1% accuracy with reasonable confidence, HABER (1970) recommends a minimum of 40,000 repetitions. If one is content with a wider confidence interval or if the variance of model results is slight, the number of repetitions can be reduced.

literature and present GQ as an attractive approach to the integration approach. Third, we present an order three, symmetric GQ.

2 Economic Simulation with Random Exogenous Variables: A Problem of Numerical Integration

A general equilibrium model, or indeed any economic simulation problem, may be viewed in the following general form:

$$G(x,a)=0 \tag{1}$$

where x represents a vector of endogenous variables and "a" represents a vector of exogenous variables (parameters, endowments, etc.). Further, define $x^*(a)$ as a solution to equation (1) and $x^*(a) \equiv H(a)$ as a vector of results of interest.

In economic models, the vector "a" typically contains estimates of behavioral parameters. In projections scenarios, the vector "a" contains estimates of future endowment levels and technical progress. These estimates are random variables. Therefore, in reality, our problem takes the form:

$$E[H(a)] = \int_{\Omega} H(a)g(a)da \tag{2}$$

for calculation of mean results, and:

$$E[(H(a)-E[H(a)])^2] = \int_{\Omega} (H(a)-E[H(a)])^2 g(a)da \tag{3}$$

for calculation of the variance of results. In equations (2) and (3), g(a) represents the multivariate density function. Calculation of higher order moments proceeds analogously.

Approaching economic simulation as a problem of numerical integration has two major advantages. First, it is more accurate than basing analysis on mean values for key exogenous variables. Second, if one has already obtained estimates of means by numerical methods, estimates of standard deviations are then easily obtained. We examine each advantage in turn.

2.1 Accuracy

In general, the expected value of a function is not equal to the value of the function evaluated at the expected value of exogenous variables. Or, more formally:

$$E[H(a)] \neq H(E[a]) \tag{4}$$

in general. The exception is the linear case. If H(·) is a linear function, equation (4) holds with equality. If H(·) can be well approximated by a linear function in the region of integration, Ω, the right hand side of equation (4) is a good approximation to the integral in equation (2). If linear approximations to H(·)

are poor, employing mean values for exogenous variables leads, in general, to significant approximation error in the estimate of the mean of results. Thus, for a non-linear model such as GTAP, simulating the model once at mean values for exogenous parameters, as is the common practice, may produce poor approximations of mean results. Of course, the extent of misrepresentation of mean results is dependent on the particular model, aggregation, and simulation employed.

To see the possibility for error in the estimation of mean values for model results, consider, as a simple example, the variable $y=e^x$, where x is distributed N(0,1). The variable y is log normally distributed. The value of the function, evaluated at the mean of the standard normal (zero), is one. However, the mean of the log normal distribution function is about 1.65. Non-linearity of the function leads to significant approximation error when estimating the mean by evaluating the function at the mean of x (zero). Economic models can also produce non-linear results over a range of reasonable values for underlying exogenous variables. For example, suppose we specify a constant elasticity demand function for an agricultural commodity. If we view supply as stochastic, the price which corresponds to the demand function evaluated at average production levels will not, in general, equal the mean price.

2.2 Standard Deviations

Once estimates of mean values for model results have been obtained by numerical methods, such as Monte Carlo or GQ, estimates of standard deviations are straightforward to obtain. This permits systematic investigation of the impact of uncertainty with respect to values of key exogenous variables under specific distributional assumptions. One can also use Chebychev's inequality to place confidence bounds on model results. With the confidence intervals in place, some results reveal themselves as robust relative to different values in the exogenous variables vector while other results reveal themselves as highly dependent on the values employed for underlying exogenous variables. This permits analysts to emphasize the most robust results.

Since numerical approximations to integration problems generally take the form of a weighted sum of evaluations of the integrand, the approach also provides a logical series of simulations. Close examination of individual simulations allows the analyst to determine how results change as the underlying exogenous variables take on different values. This aids in sharpening intuition into the functioning of the model. Finally, if results hinge on a few particularly unreliable estimates for exogenous variables (with concomitant high variance in the distribution), one may focus research resources in hopes of reducing some of the uncertainty surrounding the variables in question.

In summary, treating economic simulation with random exogenous variables as a problem of numerical integration generates more accurate results. It also provides more information about results, in the form of estimates of standard deviations on model results. In addition, the numerical integration optic shifts the debate from the best point estimate for key exogenous variables to the best distributional assumption for key exogenous variables. This is more likely to be constructive and it allows for all estimates to be considered. We turn now to an examination of some methods of numerical integration.

2.3 Numerical Integration and Gaussian Quadrature

2.3.1 The Univariate Case

Consider the univariate integration problem:

$$\int_a^b f(x)g(x)dx \tag{5}$$

where g(x) is the density function.

Even in this simple case, the integrand may be difficult or impossible to evaluate analytically. If so, one is forced to evaluate the integral numerically. In general, numerical approximations of the integral take the form:

$$\sum_{j=1}^{J} w_j \, f(x^j) \tag{6}$$

where J represents the total number of evaluations of $f(\cdot)$ and w_j represents the weight associated with each evaluation (HABER, 1970).

The Monte Carlo approach represents a special case where one generates J pseudo-random numbers from the distribution g(x) over the interval [a,b], evaluate the integrand J times, and attach a weight of 1/J to the result from each evaluation. If J is sufficiently large, the approximation will be good under extremely mild conditions on the integrand. However, in instances where the integrand is costly to evaluate, one might wish to keep the number of evaluations of the integrand, J, small. In this case, to make up for the reduced number of evaluations, one endeavors to choose points within the interval [a,b] where one evaluates the function, x^j, and associated weights, w_j, intelligently.

This problem has occupied mathematicians for some time now. As HABER states:

> "From the seventeenth century up to now, a great deal of effort went into the development of methods for approximating single integrals- such men as Gauss, Hermite, and Chebychev contributed to the subject- and a large variety of effective formulas are (sic) available." (HABER, 1970)

These formulas produce sets of points and associated weights called quadratures. GQs are especially appealing. For the case of the integration problem in equation (5), an order d GQ solves the system of equations:

$$\sum_{j=1}^{J} w_j (x^j)^s = \int_a^b (x)^s g(x)dx, \qquad s = 0,1,2,...,d \tag{7}$$

Some general comments regarding equation set (7) are useful. Note that the right hand side corresponds to the moments about zero up to order d for the density function, g(x), over the interval [a,b]. Also, note that, if g(x) is a distribution function, the weights are required to sum to one. Finally, note that there are d+1 equations in the system and that one chooses 2J pieces of information (a point and an associated weight for each element of J). In the univariate case, it turns out that one can always solve the system with J no larger than:

$$J = \left\lceil \frac{(d+1)}{2} \right\rceil \tag{8}$$

where $\lceil z \rceil$ denotes the least integer greater than or equal to z.

It is desirable that weights be positive ($w_j \geq 0$) and points lie in the domain of integration (x^j is an element of the interval [a,b]) (KRYLOV, 1962; HABER, 1970). Krylov shows that the sum of the absolute value of the weights positively impacts approximation error. Thus, positive weights keep maximum approximation error to a minimum. Haber lists two reasons why points outside the domain of integration are undesirable. First, the function may not be defined outside of the domain of integration. Second, even if the function is defined outside of the domain integration, we are interested in the behavior of the function within the domain. Points outside the domain of integration may prove misleading.

For example, in economics, points outside the domain of integration raise the specter of solving economic models with positive compensated own price elasticities of demand. The function may be defined (i.e. we might be able to solve the model); however, the results are devoid of economic meaning. On the other hand, points inside the domain are certain to give some insight. To ensure that weights are positive and points lie in the domain of integration, we must impose the mild conditions that moments up to order d are finite (e.g. the right hand sides of equation set (7) are finite) and the density function, g(x), is non-negative on the interval [a,b]. These are the only conditions necessary to assure that a GQ of order d can be drawn.

Ability to draw a GQ is not related to accuracy of approximation to the integral. Accuracy is related to the ability of a polynomial to approximate the integrand. When the integrand is a polynomial of order d or lower, GQ approximations are exact. This attribute lends GQs their special appeal. To see this, consider a simple example. Suppose $f(x)=cx^k$ where $k \leq d$, then the GQ approximation of the integral equals:

$$\begin{aligned} \sum_{j=1}^{J} w_j c(x^j)^k &= \int_a^b cx^k g(x)dx \\ c\sum_{j=1}^{J} w_j (x^j)^k &= c\int_a^b x^k g(x)dx \\ \sum_{j=1}^{J} w_j (x^j)^k &= \int_a^b x^k g(x)dx \end{aligned} \tag{9}$$

The bottom relation must hold by equation set (7). Essentially, when the integrand is a polynomial of order d or lower, coefficients cancel from the GQ approximation and the analytical solution leaving one or more of the relations set forth in equation set (7). Similarly, GQ approximations of variances will be exact when the integrand is a polynomial of order $\lfloor d/2 \rfloor$ or lower (where $\lfloor z \rfloor$ equals the greatest integer less than or equal to z).

When the integrand is not a polynomial of order d or less, the accuracy of GQ approximation depends upon two factors: the number of times the integrand may be continuously differentiated within the domain of integration,[2] ℓ, and the order of approximation, d. The first factor, ℓ (smoothness in Haber's lexicon),[3] dictates the maximum order of Taylor series approximation for an arbitrary starting point within the domain of integration. Order of approximation recognizes that some functions may be well approximated by a polynomial of order d but not so well approximated by a polynomial of order k where k<d. Thus, if the integrand is infinitely continuously differentiable, one is at liberty to choose a sufficiently high order of approximation to drive maximum approximation error below any desired threshold.

Even more attractive, the theoretical maximum approximation error declines exponentially as the order of approximation increases (HABER, 1970). For the univariate case, Haber shows that maximum approximation error to the integral in equation (5) is governed by the following relation:

$$\left| \int_a^b f(x)g(x)dx - Q(f,x^i) \right| \leq \alpha n^{-m} \tag{10}$$

where $Q(f,x^i)$ is the GQ approximation to the integral, α is a constant, n is the minimum number of points necessary for a quadrature of order d, and m (the exponential rate of decline in maximum error) equals the smaller of ℓ or d+1 (m= minimum(ℓ,d+1)).

To illustrate the gain in precision as the order of approximation increases, consider the problem:

$$\int_0^1 e^{-x} dx \tag{11}$$

In this case, the density function is g(x)=1 and is thus non-negative and has finite moments over the interval [0,1]. The integrand cannot be perfectly approximated by a polynomial of order d; however, it is infinitely continuously differentiable. The problem is thus well suited to a GQ approximation. In addition, an analytic solution to the problem exists; therefore, we may compare GQ approximations of

2. In addition, the ℓth derivative of the integrand must be continuous in the domain of integration.

3. Other authors use the term smoothness to refer to an infinitely continuously differentiable functions.

Table 1 Absolute Value of Univariate GQ Approximation Error To Equation (11)

Integrand e^{-x} on the interval [0,1]

Order of Approximation (d)	Number of Points (n)	|Error|
1	1	$6*10^{-2}$
3	2	$1.4*10^{-4}$
5	3	$7.0*10^{-7}$
7	4	$3.6*10^{-9}$
9	5	$8*10^{-11}$

Table 2 Estimated Rate of Convergence to True Values for Results in Table 1

Regression Equation: $\ln(|\text{Error}|) = \ln(\alpha) + (ß-(d+1))*\ln(n)$

Estimated Parameter ß	-3.01
Estimated Standard Deviation of ß	(0.344)
Estimated Parameter α	5.4
Regression R^2	0.96

different order to the analytic solution. Table 1 below presents the absolute value of approximation error for GQ approximations of progressively higher order.

By plugging these results into equation (10) and running a regression, one can obtain an estimate of the actual rate of convergence. In the regression equation shown at the top of table 2 below, ß and α are the parameters to be estimated. If ß<0, the rate of convergence is faster than the theoretical minimum for this particular problem. If ß=0, the rate of convergence is right at the theoretical minimum. If ß>0, the rate of convergence is slower than the theoretical minimum. Results are shown in table 2 below.

These results indicate extremely rapid convergence. For the function in question, convergence rates are on the order of $\alpha*n^{-(d+4)}$. As table 1 illustrates, each additional point yields approximately two additional

decimal points in accuracy. Furthermore, these results may not be uncommon. Haber implies that convergence rates faster than the (already fast) theoretical minimum might often occur.

A comparison to Monte Carlo methods is illuminating. It cannot be direct, however. Since Monte Carlo methods are probabilistic, there is always positive probability that the points selected might concentrate in a portion of the domain of integration resulting in substantial bias in the estimate of the integral. The upper bound on the error is thus large[4] and useless. As an alternative, Haber uses the rate of convergence of a confidence interval around the true value of the integral as the number of points employed, n, increases. This rate of convergence is $n^{-1/2}$. In other words, to obtain an additional decimal point of accuracy at a given confidence level, one must employ 100 times as many points. So, increasing the number of points from 1,000 to 100,000 results in only a one digit gain in accuracy.

When the integrand does not exhibit all of the desirable properties listed above, convergence of GQ estimates to the integral will be less rapid. For example, if the integrand can be continuously differentiated d times but not d+1 times and an order d approximation provides inadequate accuracy, increasing the order of approximation has no impact on the rate of convergence, m= minimum(ℓ,d+1). However, increasing the order of approximation does increase the required number of points and convergence continues at rate -ℓ. Discontinuous functions cannot be adequately approximated by a polynomial; thus they require other methods.

In general, results from the univariate case indicate that, for broad classes of functions, GQs offer a method to approximate definite univariate integration problems accurately while requiring a very limited number of evaluations of the integrand. Prior to the advent of the computer, this was very useful. With current computing technology, Monte Carlo methods for approximating solutions to univariate integration problems are typically quite practical. In the multi-variate case, Monte Carlo approximations are not always so practical. In fact, the growth in computing power, which has favored Monte Carlo methods for univariate integration problems, has also catalyzed the development and regular use of extraordinarily complex multivariate integrands. Consider, for example, a global general equilibrium model with substantial regional and sectoral detail. These integrands can be computationally burdensome to evaluate even with the best available technology. We turn now to the multi-variate case.

2.4 The Multivariate Case

Extension to the multivariate case is conceptually straightforward. The form of the problem remains as in equation (5) except that x is now regarded as a vector of size M and integration occurs over a region Ω. Elements of x, $[x_m]_{m=1,\ldots,M}$, are defined with a subscript. The problem thus takes the form:

$$\int_{\Omega} f(x)g(x)dx \tag{12}$$

4. The upper bound on the error is equal to the maximum of the absolute value of the difference between the true value of the integral and the value of the function evaluated at an arbitrary point within the domain of integration.

Calculation of moments becomes more involved due to cross products such as covariances. In general, a moment of order d or lower about zero (a scalar) could be written as:

$$\int_{\Omega} \left[\prod_{m=1}^{M} (x_m)^{l_m} \right] g(x) dx \tag{13}$$

where

$$l_m \in [0,1,2,...,d], \ \sum_{m=1}^{M} l_m \leq d \tag{14}$$

For example, if $l_1=1$ and all other exponents equal zero, then equation (13) gives the mean of x_1 (μ_1). Different permutations of l_m in accordance with equation (14) give all possible moments for the vector x up to order d.

The multivariate analog to equation set (7) requires that an "order d approximation" to the distribution of x satisfy the following equation:

$$\sum_{j=1}^{J} w_j \prod_{m=1}^{M} (x_m)^{l_m} = \int_{\Omega} \left[\prod_{m=1}^{M} (x_m)^{l_m} \right] g(x) dx \tag{15}$$

for all combinations of non-negative integers l_m such that:

$$\sum_{m=1}^{M} l_m \leq d \tag{16}$$

The development of an upper bound on the number of points, J, required to guarantee that a quadrature satisfying equation set (15) exists is less straightforward than in the univariate case. In an early proof, TCHAKALOFF (1957) showed that the number of points required is never greater than:

$$\binom{M+d}{d} = \frac{(M+d)*(M+d-1)*...*(M+1)}{d*(d-1)*...*(1)} = K \tag{17}$$

This number, the Tchakaloff bound, corresponds to the number of unique moments for the random variable x up to order d. In other words, the number of unique scalars described by equations (13) and (14). Recently, PRECKEL and LIU (995) appealed to the theory of linear programming to show that multivariate GQs, with strictly fewer than K points, exist for the independent case. Especially simple and efficient formulas exist for order three GQ approximations to symmetric distributions. One such formula is presented in the following section.

Most importantly, the main results from the univariate case apply directly to the multivariate case. Specifically, weights are positive and points lie in the domain of integration if: moments up to order d are finite (e.g. the right hand sides of equation set (15) are finite) and the density function, g(x), is non-negative in the region of integration, Ω. GQ approximations to multivariate integration problems will be exact when the integrand is a polynomial of order d or lower. Accuracy of the approximation depends

upon the smoothness (employing the term in the sense used by Haber) of the integrand over the domain of integration and the order of the approximation. Also, as in the univariate case, maximum approximation error declines exponentially as order of approximation increases (HABER, 1970). Thus, if the integrand is well approximated by an order d polynomial in the relevant domain, the approximation to the integral will be good.

2.5 Symmetric Order Three Gaussian Quadratures

Here, we present a method developed by STROUD (1957) for drawing order three GQs for symmetric distributions. For a model with n random exogenous variables, the method permits systematic sensitivity analysis with respect to these n exogenous variables using only 2n points or solves of the model. Recall from the introduction that conducting systematic sensitivity analysis on a model that takes five minutes to solve using 1000 Monte Carlo repetitions would take nearly 3.5 days. At five minutes per solve, the method developed by Stroud permits accurate sensitivity analysis with respect to 15 random exogenous variables in 2.5 hours (2*15*5/60=2.5). If results can be well approximated by an order three polynomial, the GQ sensitivity analysis will be very accurate despite the limited number of evaluations of the model.

One might reasonably ask if order three is sufficient with respect to GTAP. For this technical paper, a comparison of results for order three approximations with results from order nine approximations was conducted. Protection levels for one commodity in one region were shocked in a three good three country aggregation of the GTAP model in which substitution parameters were treated as uncertain (ARNDT and HERTEL). The results in this case indicated that order three approximations are quite good. In all cases, the order three and order nine approximations matched to four decimal points. In most instances, they matched to six decimal points. Doubtless, the quality of an order three approximation will not be so good for all cases of systematic sensitivity analysis (univariate and multi-variate). In some instances, the quality of approximation might deteriorate dramatically.

Again, a comparison to Monte Carlo methods is illuminating. Haber shows that the standard deviation of a Monte Carlo approximation to the mean of a result is proportional to the true standard deviation of the result and inversely proportional to the square root of the number of repetitions. More formally,

$$\sigma^* = n^{-1/2}\sigma \tag{18}$$

where σ^* is the standard deviation of the Monte Carlo approximation, n is the number of repetitions, and σ is the true standard deviation of the result. Thus, if the true standard deviation of the targeted result, σ, is three, the standard deviation of result, σ^*, is 0.095 given 1000 Monte Carlo repetitions. Thus, in this instance, Monte Carlo does not even assure accuracy to one decimal point with reasonable confidence.

Below, we present the formula developed by Stroud for deriving equally weighted, order three quadratures for symmetric, independent distributions of mean zero and standard deviation one. Let n be the number of random exogenous variables. Let Γ_k $(\gamma_{k1},\gamma_{k2},...,\gamma_{kn})$ be the k^{th} quadrature point

(k=1,2,...,2n). Let r=1,2,...,⌊n/2⌋ where ⌊n/2⌋ denotes the greatest integer not exceeding n/2. Points may be derived by the following formula:

$$\gamma_{2r-1}=\sqrt{2}\cos\left(\frac{(2r-1)k\pi}{n}\right) \qquad \gamma_{2r}=\sqrt{2}\sin\left(\frac{(2r-1)k\pi}{n}\right) \tag{19}$$

If one is an odd number, then $\gamma_{kn}=(-1)^k$. Since weights, w_k, are equal and must sum to one, then $w_k=1/2n$. Stroud proves that points derived from the above formula satisfy the conditions set forth in equation set (15).

The presentation is fully general. Suppose we wish to conduct sensitivity analysis with respect to a symmetrically distributed random variable x ($x_1,x_2,...,x_n$), a column vector of size n with mean μ and variance covariance matrix Σ. If Σ is diagonal, the desired quadrature, Φ, can be obtained by:

$$\Phi=\mu+\Gamma\sqrt{\Sigma} \tag{20}$$

If Σ is not diagonal, a diagonal matrix, D, can be obtained via a Cholesky factorization, $\Sigma=LDL^t$. The quadrature from equation (19) is transformed to $\Gamma^*=\Gamma L$ and the desired quadrature Φ can be obtained by:

$$\Phi=\mu+\Gamma^*\sqrt{D} \tag{21}$$

The symmetric case is especially well suited to evaluation by GQ. Note that the number of points required, 2n, falls well below the Tchakaloff bound. For the case of 15 random variables, the Tchakaloff bound is 816 points (see equation (17)) while the Stroud formula requires only 30 points. The quadrature developed by Stroud takes advantage of numerous dependencies in the system of equations described in equation set (15). The Stroud formula for obtaining order three symmetric quadratures is also extremely simple and easy to use.

Methods for obtaining GQs for asymmetric distributions or for orders of approximation greater than three (d >3) are described in PRECKEl and LIU (1995). Currently, the cost of obtaining quadratures in these cases is non-trivial. To obtain a quadrature with the number of points equal to the Tchakaloff bound requires solving a linear program. Further reduction in the number of points requires solving a non-linear program. If the number of random variables exceeds 10, the programming problem become quite large even for a relatively low order approximation (d=3). Thus, the Stroud formula currently offers the simplest means for conducting systematic sensitivity analysis in large models.

3 Conclusion

Gaussian quadratures provide a practical means to conduct systematic sensitivity analysis in large models. This is especially true in the case where random exogenous variables are distributed symmetrically and model results are reasonably well approximated by an order three polynomial. In this case, one can easily obtain quadratures using the Stroud formula presented above. Outputs include good approximations of means of model results and associated standard deviations. As opposed to simply

running the model at mean values for random exogenous variables, outputs from the systematic sensitivity analysis procedure are more correct and provide rigorous insight into robustness of model results. Compared with Monte Carlo, Gaussian quadratures provide good approximations while using dramatically fewer points.

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