Elasticities of Substitution in Nested CES Systems

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Burniaux and Truong [1] cite several times (on pages 18, 20, 24, and 30) a formula from Keller ([3], page 83, equation (5.67)) for elasticities of substitution in nested CES systems. Unfortunately, the presentation in Burniaux and Truong [1] is not quite clear or consistent, while that in Keller ([3], page 83) employs notation defined over many preceding pages. Keller [3] in turn cites Keller [2], which is briefer, but still contains a good deal of material not to the immediate purpose. This note provides a minimal statement and derivation of the Keller formula.

Our approach differs from Keller [2] in that he deals with nested origin-shifted CES systems, while we consider only nested CES systems strictly defined, and that he derives the demand equations in the levels, while we provide only a differential solution. Our notation is somewhat different from and, we hope, clearer than that of Keller [3].

In a nested CES production system, we have *commodities* comprising *initial inputs*, *intermediate aggregates*, and a *final output*. Each intermediate aggregate is both an *intermediate input* and an *intermediate output*. In each nest, the output is a *direct product* of the inputs, and the inputs are *direct factors* of the output. There are also *indirect products* and *indirect factors*: a product of a product of an input is an indirect product of that input, and a factor of a factor of an output is an indirect factor of that output.

Let $\mathcal{C}$ denote the set of commodities, $\mathcal{O}$ the set of outputs, both final and intermediate, and $\mathcal{I}$ the set of inputs, both intermediate and initial. Let $\pi : \mathcal{I} \to \mathcal{O}$ map each input to its direct product. Let $\mathcal{N}$ denote the set of initial inputs, and f, final output.

For each commodity i, let P_i denote price, Q_i quantity demanded, and p_i and q_i the corresponding log differentials, $p_i = d \log P_i$ and $q_i = d \log Q_i$. Let V_i denote cost, $V_i = P_i Q_i$. For each input i, let S_i denote its share in the cost of its direct product, $S_i = V_i / V_{\pi(i)}$. For each output i, let σ_i denote the elasticity of substitution between its direct factors. Then the demand system generated by the production system is the system of equations

$$p_i = \sum_{j \in \mathcal{I}: \pi(j) = i} S_j p_j, \qquad i \in \mathcal{O}, \tag{1}$$

$$q_i = q_{\pi(i)} - \sigma_{\pi(i)} \left(p_i - p_{\pi(i)} \right), \qquad i \in \mathcal{I}. \tag{2}$$

We take as given the prices $p_i,\ i \in \mathcal{N}$ of initial inputs and the quantity q_f of final output, and solve for the prices $p_i,\ i \in \mathcal{O}$ of outputs and the quantities $q_i,\ i \in \mathcal{I}$ of inputs.

The *depth* of a commodity is the number of direct factor-product relations between it and final output, so that final output itself has depth 0, its direct factors have depth 1, and so on. Let $d : \mathcal{C} \to \mathbf{N}$ map each commodity i to its depth $d(i)$, where $\mathbf{N}$ denotes the set of natural numbers, zero-inclusive. The *depth of the system* is the maximum of the depths of the commodities, $\max\{n \in \mathbf{N} : \text{for some } i \text{ in } \mathcal{C},\ d(i) = n\}$.

For n in $\mathbf{N}$, let $\mathcal{E}_n$ denote the set of commodities of depth n, and $\mathcal{B}_n$ the set of commodities of depth greater than or equal to n. For n in $\mathbf{N}$, let $\pi_n : \mathcal{B}_n \to \mathcal{E}_n$ map each commodity in $\mathcal{B}_n$ to its unique product in $\mathcal{E}_n$, that is, to its level-n product. For i in $\mathcal{E}_n$, we set $\pi_n(i) = i$, that is, we call each commodity (degenerately) its own product at its own level.

For n in $\mathbf{N}$ and i in $\mathcal{B}_n$, let $S_{n,i}$ denote the share of commodity i in the cost of its level-n product $\pi_n(i)$, $S_{n,i} = V_i / V_{\pi_n(i)}$. Special cases are, for $n = d(i)$, $S_{n,i} = 1$, and for $n = d(i) - 1$, $S_{n,i} = S_i$. More generally, the share of a commodity in a product's products' costs is equal to the product of its share in the earlier product's costs and the share of the earlier in the later product's costs:

Proposition 1 *For i in $\mathcal{C}$ and m and n in $\mathbf{N}$ such that $n \le m \le d(i)$, $S_{n,i} = S_{n,\pi_m(i)} S_{m,i}$.*

Let $\Phi : \mathcal{C} \to \mathcal{N}$ map each commodity to the set of its factors in the set of initial inputs, $\Phi(i) = \left\{ j \in \mathcal{N} : \pi_{d(i)}(j) = i \right\}$. Let $\phi : \mathcal{C} \times \mathcal{C} \to \{0, 1\}$ be the truth function for the product-factor relation, with $\phi(i, j)$ equal to 1 if j is a factor of i, and 0 otherwise.

Let $\kappa : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ map each pair of commodities to its first common product, and $c : \mathcal{C} \times \mathcal{C} \to \mathbf{N}$ map to that product's depth, $c(i, j) = d(\kappa(i, j))$.

For n in $\mathbf{N}$ and i in $\mathcal{B}_i$ such that $\pi_n(i) \in \mathcal{O}$ (that is, such that the level-n product of i is an output, equivalently, such that i is not an initial input of depth n), let $\sigma_{n,i}$ denote the elasticity of substitution between factors of the level-n product of commodity i, $\sigma_{n,i} = \sigma_{\pi_n(i)}$.

Proposition 2 *For the nested* CES *demand system* (1)–(2),

$$p_i = \sum_{j \in \Phi(i)} S_{d(i),j} p_j, \qquad i \in \mathcal{O}$$

$$q_i = q_f + \sum_{j \in \mathcal{N}} \Big[\sum_{n=0}^{c(i,j)-1} (S_{n,j} - S_{n+1,j})\, \sigma_{n,\kappa(i,j)} + (1 - \phi(i,j))\, S_{c(i,j),j} \sigma_{\kappa(i,j)} \Big] p_j, \quad i \in \mathcal{I}$$

Proof is by induction on the depth of the system, and proposition 1.

Our object is to obtain the elasticity of substitution, $(1/S_{0,j})(\partial \log Q_i / \partial \log P_j)$, for distinct initial inputs i and j. This is just the product of $(1/S_{0,j})$ and the coefficient of p_j in the equation for q_i in proposition 2.

For n in $\mathbf{N}$ and j in $\mathcal{B}_n$, let $T_{n,j}$ denote the share of the level-n product of j in the cost of final output, $T_{n,j} = S_{0,\pi_n(j)}$. Then, for all j in $\mathcal{C}$, $T_{d(j),j} = S_{0,j}$ and $T_{0,j} = 1$.

Proposition 3 *For the nested* CES *demand system* (1)–(2), *for distinct initial inputs i and j, the elasticity of substitution between i and j is*

$$\sum_{n=0}^{c(i,j)-1} \left(\frac{1}{T_{n,\kappa(i,j)}} - \frac{1}{T_{n+1,\kappa(i,j)}} \right) \sigma_{n,\kappa(i,j)} + \frac{\sigma_{c(i,j),\kappa(i,j)}}{T_{c(i,j),\kappa(i,j)}}$$

or (rewriting the last term)

$$\sum_{n=0}^{c(i,j)-1} \left(\frac{1}{T_{n,\kappa(i,j)}} - \frac{1}{T_{n+1,\kappa(i,j)}} \right) \sigma_{n,\kappa(i,j)} + \frac{\sigma_{\kappa(i,j)}}{S_{0,\kappa(i,j)}}.$$

Proof is by propositions 2 and 1 and the definition of elasticity of substitution.

References

[1] Jean-Marc Burniaux and Truong P. Truong. GTAP-E: An energy-environmental version of the GTAP model. Technical Report 16, Center for Global Trade Analysis, January 2002.

[2] Wouter J. Keller. A nested CES-type utility function and its demand and price-index functions. *European Economic Review*, 7:175–186, 1976.

[3] Wouter J. Keller. *Tax Incidence: A General Equilibrium Approach*. North-Holland, 1980.