Elasticities of Substitution in Nested CES Systems

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Burniaux and Truong [1] cite several times (on pages 18, 20, 24, and 30) a formula from Keller ([3], page 83, equation (5.67)) for elasticities of substitution in nested CES systems. Unfortunately, the presentation in Burniaux and Truong [1] is not quite clear or consistent, while that in Keller ([3], page 83) employs notation defined over many preceding pages. Keller [3] in turn cites Keller [2], which is briefer, but still contains a good deal of material not to the immediate purpose. This note provides a minimal statement and derivation of the Keller formula.

Our approach differs from Keller [2] in that he deals with nested origin-shifted CES systems, while we consider only nested CES systems strictly defined, and that he derives the demand equations in the levels, while we provide only a differential solution. Our notation is somewhat different from and, we hope, clearer than that of Keller [3].

In a nested CES production system, we have commodities comprising initial inputs, intermediate aggregates, and a final output. Each intermediate aggregate is both an intermediate input and an intermediate output. In each nest, the output is a direct product of the inputs, and the inputs are direct factors of the output. There are also indirect products and indirect factors: a product of a product of an input is an indirect product of that input, and a factor of a factor of an output is an indirect factor of that output.

Let $C$ denote the set of commodities, $O$ the set of outputs, both final and intermediate, and $I$ the set of inputs, both intermediate and initial. Let $\pi : I \to O$ map each input to its direct product. Let $N$ denote the set of initial inputs, and $f$, final output.

For each commodity $i$, let $P_i$ denote price, $Q_i$ quantity demanded, and $p_i$ and $q_i$ the corresponding log differentials, $p_i = d \log P_i$ and $q_i = d \log Q_i$. Let $V_i$ denote cost, $V_i = P_i Q_i$. For each input $i$, let $S_i$ denote its share in the cost of its direct product, $S_i = V_i / V_{\pi(i)}$. For each output $i$, let $\sigma_i$ denote the elasticity of substitution between its direct factors. Then the demand system generated by the production system is the...
system of equations

\[ p_i = \sum_{j \in \mathcal{I} : \pi(j) = i} S_j p_j, \quad i \in \mathcal{O}, \quad (1) \]

\[ q_i = q_{\pi(i)} - \sigma_{\pi(i)} \left( p_i - p_{\pi(i)} \right), \quad i \in \mathcal{I}. \quad (2) \]

We take as given the prices \( p_i, i \in \mathcal{N} \) of initial inputs and the quantity \( q_f \) of final output, and solve for the prices \( p_i, i \in \mathcal{O} \) of outputs and the quantities \( q_i, i \in \mathcal{I} \) of inputs.

The depth of a commodity is the number of direct factor-product relations between it and final output, so that final output itself has depth 0, its direct factors have depth 1, and so on. Let \( d : \mathcal{C} \to \mathcal{N} \) map each commodity \( i \) to its depth \( d(i) \), where \( \mathcal{N} \) denotes the set of natural numbers, zero-inclusive. The depth of the system is the maximum of the depths of the commodities, \( \max\{ n \in \mathcal{N} : \text{for some } i \in \mathcal{C}, d(i) = n \} \).

For \( n \in \mathcal{N} \), let \( \mathcal{E}_n \) denote the set of commodities of depth \( n \), and \( \mathcal{B}_n \) the set of commodities of depth greater than or equal to \( n \). For \( n \in \mathcal{N} \), let \( \pi_n : \mathcal{B}_n \to \mathcal{E}_n \) map each commodity in \( \mathcal{B}_n \) to its unique product in \( \mathcal{E}_n \), that is, to its level-\( n \) product. For \( i \) in \( \mathcal{E}_n \), we set \( \pi_n(i) = i \), that is, we call each commodity (degenerately) its own product at its own level.

For \( n \in \mathcal{N} \) and \( i \) in \( \mathcal{B}_n \) such that \( \pi_n(i) \in \mathcal{O} \) (that is, such that the level-\( n \) product of \( i \) is an output, equivalently, such that \( i \) is not an initial input of depth \( n \)), let \( \sigma_{n,i} \) denote the elasticity of substitution between factors of the level-\( n \) product of commodity \( i \), \( \sigma_{n,i} = \sigma_{\pi_n(i)} \).

**Proposition 1** For \( i \) in \( \mathcal{C} \) and \( m \) and \( n \) in \( \mathcal{N} \) such that \( n \leq m \leq d(i) \), \( S_{n,i} = S_{n,\pi_m(i)} S_{m,i} \).
Proposition 2  For the nested CES demand system (1)–(2),

\[ p_i = \sum_{j \in \Phi(i)} S_{d(i),j} p_j, \quad i \in O \]

\[ q_i = q_f + \sum_{j \in N} \left[ \sum_{n=0}^{c(i,j) - 1} (S_{n,j} - S_{n+1,j}) \sigma_{n,\kappa(i,j)} + (1 - \phi(i,j)) S_{c(i,j),j} \sigma_{\kappa(i,j)} \right] p_j, \quad i \in I \]

Proof is by induction on the depth of the system, and proposition 1.

Our object is to obtain the elasticity of substitution, \((1/S_{0,j})(\partial \log Q_i/\partial \log P_j)\), for distinct initial inputs \(i\) and \(j\). This is just the product of \((1/S_{0,j})\) and the coefficient of \(p_j\) in the equation for \(q_i\) in proposition 2.

For \(n\) in \(N\) and \(j\) in \(B_n\), let \(T_{n,j}\) denote the share of the level-\(n\) product of \(j\) in the cost of final output, \(T_{n,j} = S_{0,\pi_n(j)}\). Then, for all \(j\) in \(C\), \(T_{d(j),j} = S_{0,j}\) and \(T_{0,j} = 1\).

Proposition 3  For the nested CES demand system (1)–(2), for distinct initial inputs \(i\) and \(j\), the elasticity of substitution between \(i\) and \(j\) is

\[ \sum_{n=0}^{c(i,j) - 1} \left( \frac{1}{T_{n,\kappa(i,j)}} - \frac{1}{T_{n+1,\kappa(i,j)}} \right) \sigma_{n,\kappa(i,j)} + \frac{\sigma_{c(i,j),\kappa(i,j)}}{T_{c(i,j),\kappa(i,j)}}, \]

or (rewriting the last term)

\[ \sum_{n=0}^{c(i,j) - 1} \left( \frac{1}{T_{n,\kappa(i,j)}} - \frac{1}{T_{n+1,\kappa(i,j)}} \right) \sigma_{n,\kappa(i,j)} + \frac{\sigma_{\kappa(i,j)}}{S_{0,\kappa(i,j)}}. \]

Proof is by propositions 2 and 1 and the definition of elasticity of substitution.

References

