Mathematics of Generalized Versions of the Melitz, Krugman and Armington Models with Detailed Derivations

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Two Objectives of this paper

- 1. Document the mathematics of the Balistreri and Tarr (2020) models

a comprehensive evaluation (18 model variants) of the estimated relative welfare gains from changes in trade costs in a Melitz (2003) based model compared with models based on Armington (1969) and Krugman (1980).

2. provide pedagogical mathematical derivations of the models that will make it easier to learn the models, especially the heterogeneous firms model of international trade

Our Full Policy Model Extends the Melitz (2003) Version of the Heterogeneous Firms Model of International Trade, as follows

- Multiple Sectors
- Intermediates included (with shares based on GTAP input-output data)
- Labor-leisure choice
- Heterogeneous regions
- Multiple primary factors of production
- Sector-specific factors of production
- Non-zero trade balances between regions
- Tariffs and tariff policies included, not just iceberg trade costs,
- Both global and unilateral tariff policies modeled

These extensions are either necessary or potentially useful for empirical CGE applications. Our empirical model uses GTAP data.

Claim: despite its generality, much easier to learn the Heterogeneous Firms model with this paper

- Especially with the heterogeneous firms model, we take a pedagogical approach.
- Detailed step by step derivations
- Feedback we have received to date supports our claim
- We hope this will be a clear roadmap for understanding the theory of modern multi-sector, multi-region international trade models that must be fitted to data.

Literature on the Mathematics of the Heterogeneous Firms model

- **Redding (2010a; 2010b)** provides a good mathematical explanation of the model of Melitz (2003). See also **Donaldson (2016)**. But we need to extend that model for an empirical application.

- **Balistreri and Rutherford (2013)** complements this paper

Provides calibration and computer code for our model.

But it lacks details of the derivations of the equations.

Literature on the Mathematics of the Heterogeneous Firms model (2)

- **Akgul, Villoria and Hertel (2016, JGEA)**.
- Provides mathematics of a generalized heterogeneous-firms model and computer code for how to implement a heterogeneous-firms model in GTAP
- excellent explanation of the economic intuition for the equations of the heterogeneous-firms model.
- But no bilateral selection into export markets which is a hallmark of the Melitz approach to explain different export shares by destination market.

Dixon, Jerie and Rimmer (2018). A comprehensive approach to the Armington, Krugman and Melitz models—mathematics, computer code in GEMPACK, calibration and economic theory.

Regarding the math, we believe our approach adds value because:

1. We use continuous variables—essential for reading the theory literature.
2. We provide the **derivations** with intermediates and multiple primary factors.
3. We include labor-leisure choice in the model with derivations
4. Greater generality of the pdf. As in Melitz (2003), we derive the price and quantity of the representative firm, industry revenue and industry profits, Dixit-Stiglitz price and quantity with general probability functions.

The Pareto distribution is assumed only to obtain the bilateral cutoffs and zero profit condition, which depend on the pdf. Useful for those who wish to use a pdf other than Pareto, such as Fernandes *et al*. (2020).

There is a core set of equations that are common to all three market structures

- Our mathematics shows that there is a set of equations that are common to the Armington, Krugman and Melitz models.
- Shown by Balistreri and Rutherford (2013). Elaborated by Dixon, Jerie and Rimmer (2018)
- The difference in these common equations is in the interpretation of some of the variables.

We provide 4 tables of Model Equations and Associated Variables

1. Equations common to all models (7 sets)
2. Armington specific equations (2 sets)
3. Krugman specific equations (5 sets)
4. Melitz specific equations (7 sets)

Prerequisites

- Differential and Integral Calculus of many variables
- Basic Probability Theory
- Knowledge of microeconomic theory, particularly duality theory and Shephard's Lemma.

Example of a Common Equation

Composite Input in all models

We specify technology in the dual--combines the technology and the optimization.

We assume there is a composite input that is a linear homogeneous function of all intermediates and primary factors;

Primary factors combine in a Cobb-Douglas nest and may be fully mobile or sector-specific.

We allow intermediates to take various linear homogeneous forms—.

c_{ir} is the cost of the one unit of composite input in sector *i* of region *r;* it has the following form if intermediates are CES and they combine with value-added in a CES nest:

$$c_{ir} = \left[\sum_{j \in I} \alpha_{jir} P_{jr}^{1-\sigma^T} + \alpha_{wir} \left(\prod_{f \notin \tilde{F}} (w_{fr})^{\beta_{fir}} \prod_{f \in \tilde{F}} (\tilde{w}_{fir})^{\beta_{fir}} \right)^{1-\sigma^T} \right]^{1/(1-\sigma^T)} \quad i \in I,\ r \in R$$

$w_{jr}\,(\tilde{w}_{jr})$ is the price of mobile (sector-specific) primary factor *j* in region *r*.

P_{jr} is the price of intermediates in sector *j* or region *r*.

Interpretation of the price of intermediates varies with the model

$$c_{ir} = \left[\sum_{j \in I} \alpha_{jir} P_{jr}^{1-\sigma^T} + \alpha_{wir} \left(\prod_{f \notin \tilde{F}} (w_{fr})^{\beta_{fir}} \prod_{f \in \tilde{F}} (\tilde{w}_{fir})^{\beta_{fir}} \right)^{1-\sigma^T} \right]^{1/(1-\sigma^T)} \quad i \in I,\ r \in R$$

P_{jr} is the Armington aggregate in the Armington model

P_{jr} is the Dixit-Stiglitz aggregate in the two monopolistic competition models.

Monopolistic Competition Models—There are fixed costs of operation and, in Melitz, fixed entry costs

We assume that variable costs and all fixed costs use the same bundle of composite inputs.

This follows the approach of: Helpman and Krugman (1985, p. 12), Costinot and Rodriguez-Clare (2013, equation 7), Balistreri and Rutherford (2013, equation 23.2), Dixon *et al*. (2019, equation 2) and Markusen, Rutherford and Tarr (2005, equations 5 and 6)

Since firms use the composite input, the math is no more complicated than the various theory models that assume labor is the sole factor of production.

Balistreri and Tarr (2020): welfare comparison of trade costs in Armington, Krugman and Melitz models—very brief summary of key results

18 model or policy variants--v

To achieve transparency, we start with small, simplified models and add model complexity, sequentially, isolating impacts of new features.

Trade Responses are Held Equal.

In all models, we adjust elasticities in the Armington, Krugman and Melitz models such that the change in the trade flows (the trade responses) are held equal in response to a change in trade costs.

We use three calibration strategies to achieve equal trade responses, and show the results are robust to the calibration strategies.

Define:

M **as the *global* change in welfare wrt a *global* change in trade costs in the Melitz model;**

K **as the *global* change in welfare wrt a *global* change in trade costs in the Krugman model;**

A **as the *global* change in welfare wrt a *global* change in trade costs in the Armington model.**

Table 6: Summary of Relative Welfare Gains in the Melitz, Krugman and Armington Models
Welfare Gains are Global Equivalent Variation as a Percent of Consumption from change in trade costs (except for row 18).

Armington model welfare gains are indexed at one.	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
All models are global with ten heterogeneous regions.	**New or Known Result**	**Trade Responses Equalized**			**Krugman's share of Melitz's gains above Armington***	**Trade Responses NOT Equalized**		
		Relative Welfare Gains				**Relative Welfare Gains**		
Model Assumption and Policy Scenario		**Armington**	**Krugman**	**Melitz**		**Armington**	**Krugman**	**Melitz**
I. Global 10% Reduction in Iceberg Trade Costs		σ^A adjusted	σ^K adjusted	σ^M=5.0; a=4.58	***S = (K-A)/(M-A)***	σ^A = 5.58	σ^K = 5.58	σ^M=5.0; a=4.58
A. One-sector Model			*K/A*	*M/A*			*K/A*	*M/A*
1. stylized model features**	Known	1	1	1	1	1	1	1
2. stylized model with trade imbalances***	New	1	1	1	1	1	1	1
3. stylized with labor-leisure choice	New	1	1.06	1.07	0.87	1	1.06	1.07
4. stylized with an intermediate good	Known	1	1.35	1.43	0.82	1	1.35	1.43
5. stylized with an intermediate good and labor-leisure choice	New	1	1.61	1.79	0.78	1	1.61	1.79
B. Four-Sector Model (one primary factor, no labor-leisure choice)								
6. with one aggregate intermediate good	Known	1	1.38	1.47	0.80	1	1.40	1.49
7. with Cobb-Douglas demand for 4 intermediate goods	New	1	1.55	1.72	0.76	1	1.59	1.76
8. with CES demand for 4 intermediates and elas. of sub. = 0.5	New	1	1.20	1.24	0.84	1	1.22	1.26
C. Four-Sector Model (includes intermediates with esub = 0.5 and tariff data)								
9. with 1 mobile primary factor	New	1	1.18	1.22	0.83	1	1.21	1.25
10. with 3 mobile primary factors	New	1	1.18	1.21	0.83	1	1.21	1.24
11. with 3 primary factors with one of them (capital) 20% sector-specific	New	1	1.16	1.20	0.83	1	1.18	1.21
12. with 3 mobile primary factors and labor-leisure choice	New	1	1.26	1.32	0.82	1	1.28	1.34
D. Policy Model(see footnote **)**								
13. Policy Model except no labor-leisure choice	New	1	1.12	1.13	0.88	1	1.15	1.16
14. Policy Model	New	1	1.17	1.20	0.87	1	1.20	1.23
II. Tariffs Changes: Movement to Global Free Trade								
15. Policy Model except no labor-leisure choice	New	1	1.66	1.69	0.95	1	2.55	2.63
16. Policiy Model starting from uniform tariffs, no labor-leisure choice	New	1	4.06	4.67	0.83	1	5.21	5.76
17. Policy Model	New	1	1.97	2.08	0.90	1	3.08	3.26
III. Uniltaral Increases in All Tariffs to 25%		all ten regions positive	six out of ten regions negative	seven out of ten regions negative		all ten regions positive	six out of ten regions negative	seven out of ten regions negative
18. Policy Model **Sign of the Welfare Change**	New							

Footnotes for the Table.

*Krugman model's share of the larger welfare gains of the Melitz model over Armington.

**Stylized features are those of Arkolakis *et al.* (2012) and are: one sector, one fully mobile primary factor; no intermediates; no labor-leisure choice; no initial tariffs; and balanced trade in all regions.

***All additional models contain data-based trade imbalances.

****Policy Model includes nine sectors, labor-leisure choice, one sector-specific and two mobile primary factors, initial data-based tariffs and trade balances, CES demand for intermediates with data-based shares and elasticity of substitution of 0.5 and ten heterogeneous regions; the monopolistic competition models contain four Armington and five monopolistically competitive sectors.

$M > K > A$

In all the 15 model variants with global changes in iceberg costs or tariffs, other than a stylized one-sector model, we find:

Depending on the model, we have:

$(1.07)A \leq M \leq (4.67)A$ and

$(1.06)A \leq K \leq (4.06)A$.

The Krugman model captures between 76% to 95% of the gains above Armington that are captured by the Melitz model. Thus, the welfare results of the Krugman model are much closer to the Melitz model than Armington.

This suggests that the variety effect is quantitatively more important than the selection effect in differentiating the monopolistic competition models from Armington.

Holding trade responses equal reduces the advantage of the monopolistic competition models over Armington.

For individual regions, in multi-sector models we find cases of reversed welfare rankings. Larger terms-of-trade gains in the Armington model for these regions almost always explain these cases.

Intermediates and Labor-Leisure Choice are the key model features that separate the welfare results across market structures

(i) **Intermediates** significantly increase the gains in the Melitz and Krugman models compared to Armington;

(ii) **Labor-leisure choice** also increases the gains in the monopolistic competition models compared to Armington;

(iii) **Interaction effect** between intermediates and labor-leisure choice magnifies the impact of labor-leisure choice on the relative gains of the Melitz and Krugman models compared to Armington (more than tripled in our application)

Policies—Much lower optimal tariff in the monopolistic competition models

We simulate a unilateral *increase in tariffs* to 25 percent (without retaliation) separately for each of the ten regions of our full policy model.

all ten of our regions gain in the Armington model, but six of the ten regions lose in the Krugman model and seven of the regions lose in the Melitz model.

for the majority of the ten regions the sign of the welfare effect is opposite.

Why

Armington model: well-known strong terms-of-trade gains from the unilateral imposition of tariffs up to the optimal tariff. Given our adjustment of elasticities based on structural gravity of 4.6, the optimal tariff for these countries is about 21-23 percent.

Monopolistically competitive sectors: losses from the product variety externality for even small increases in tariffs. This works in the opposite direction of the terms-of-trade gain.