Empirical estimates of the elasticity of substitution of a KLEM production function without nesting constraints: The case of the Variable Output Elasticity Cobb-Douglas

Paul Malliet (OFCE-Sciences Po, Paris Nanterre University)
Frédéric Reynès (OFCE-Sciences Po, NEO, TNO)

GTAP Conference 2022

Motivation

- Computable General Equilibrium (CGE) models rely on the extensive use of national accounting data and input-output tables.

- Less attention is paid on the value of exogenous parameters such as elasticities
  - Use of aggregate estimation of elasticity parameters
  - Alternatively, estimation from micro-data on specific sectors

- The distribution of the parameters values plays a central role in the results obtained from simulations (Jacoby & al, 2006; Jaccard & Bataille, 2008; Okagawa & Ban, 2008)

- Recent development on multi-regional input databases provides more deepth in the data to perform econometric estimations

Problematic and research question

- How the specification of the production function imposes estimation results and what form should be prefered on another ?
- The Constant Elastictity of Substitution (CES) Production function has the advantage of tractability but has also drawbacks:
  - Constraints imposed on the distribution of the parameters when the number of inputs is greater than 2 (Uzawa, 1962; McFadden, 1963)
  - The nesting of the production process allows for overcoming this limit (Sato,1967)
- More flexible function (Translog) are difficult to implement in a CGE framework
- What functional form should be prefered for econometric estimations?
- Which one should we use in a CGE framework?
  - A question at the start of the ThreeME model
  - Objective: find a more general function than the CES function but more tractable function than the Translog

Estimation of a new flexible production function

- Reynès F. (2019), The Cobb-Douglas function as a flexible function. A new perspective on homogeneous functions through the lens of output elasticities, Mathematical Social Sciences 97 (2019), 11–17.

- Advantages compared to standard flexible functions such as the Translog function:
  - [1] It avoids the tedious algebraic of the second order approximation
  - [2] This greatly facilitates the deduction of linear input demands function without the need of involving the duality theorem and the approximation of the cost function at the optimum.
  - [3] It allows for a tractable generalization of the CES function to the case where the elasticity of substitution between each pair of inputs is not necessarily the same
  - [4] This provides a more general and more flexible framework compared to the traditional nested CES approach while facilitating the analyze of the substitution properties of nested CES functions.

- Example of the case of substitutions between energy, capital and labor

The **Variable** Output Elasticities Cobb-Douglas function (VOE-CD)

- **DEFINITION 1.** The production function is:
  - 1. a continuous and twice differentiable function Q: $Q = Q(X_1, X_2, ..., X_i, ..., X_n)$
  - 2. homogeneous of degree 1 (**constant returns-to-scale**)
  - 3. increasing in inputs: $Q'(X_i) = \frac{\partial Q}{\partial X_i} > 0$
  - 4. strictly concave (reflecting the **law of diminishing marginal returns**): $Q''(X_i) = \frac{\partial^2 Q}{(\partial X_i)^2} < 0$

$$\frac{\partial\left(Q'(X_i)\right)}{\partial X_j} = \frac{\partial^2 Q}{\partial X_i \partial X_j} > 0$$

- **PROPOSITION 1.** Any homogeneous production function as defined in Definition 1 can be written as follows:

$$\boxed{\dot{Q} = \sum_{i=1}^{n} \varphi_i \dot{X}_i \Leftrightarrow \mathrm{d}(\ln Q) = \sum_{i=1}^{n} \varphi_i \, \mathrm{d}(\ln X_i)}$$

Cobb-Douglas function

$$\varphi_i = \frac{Q'(X_i)X_i}{\sum_{j=1}^{n} Q'(X_j)X_j} = \left[\sum_{j=1}^{n}\left(\frac{Q'(X_j)X_j}{Q'(X_i)X_i}\right)\right]^{-1}$$

With:
Variable Output Elasticities

- If the Output Elasticities (OE) are constant, traditional Cobb-Douglas function:

$$Q = \prod_{i=1}^{n} X_i^{\varphi_i} \Leftrightarrow \ln Q = \sum_{i=1}^{n} \varphi_i \ln X_i$$

The VOE-CD function as a generalization of the CES function

- **DEFINITION 3.** The Elasticity of Substitution (ES) of Hicks (1932) and Robinson (1933) between inputs i and j measures the change in the ratio between two factors of production due to a change in their relative marginal productivity, i.e. in the Marginal Rate of Substitution (MRS). Its specification is:

$$-\eta_{ij} = \frac{\mathrm{d}\ln(X_i / X_j)}{\mathrm{d}\ln\left(Q'(X_i)/Q'(X_j)\right)} \Leftrightarrow \frac{Q'(X_i)}{Q'(X_j)} = \frac{\tilde{\varphi}_i}{\tilde{\varphi}_j}\left(\frac{X_i}{X_j}\right)^{-1/\eta_{ij}}$$

$$\Leftrightarrow \dot{X}_i - \dot{X}_j = -\eta_{ij}\left(\dot{Q}'(X_i) - \dot{Q}'(X_j)\right)$$

- **PROPOSITION 3.** The combination of the definition of the ES (Definition 3) to the definition of the VOE-CD function (Definition 2.1) provides a generalization of the CES function where the ES between each pair of inputs are not necessarily the same and where the OE of input j is:

$$\varphi_i = \left[\sum_{j=1}^{n}\left(\frac{\tilde{\varphi}_j}{\tilde{\varphi}_i}\left(\frac{X_j}{X_i}\right)^{1-1/\eta_{ij}}\right)\right]^{-1}$$

OE depend on
- **the ES between pair of inputs**
- **the input ratios**

The VOE-CD function as a generalization of the CES function

- Minimization of production cost subject to the technical constraint (VOE-CD):

$$C = \sum_{i=1}^{n} P_i^X X_i$$

- The well-known first order necessary conditions say that at the optimum, the ratio between the marginal productivities of two inputs equals the ratio between their prices:

$$\mathcal{Q}'(X_i) / \mathcal{Q}'(X_j) = P_i^X / P_j^X$$

- At the optimum, the OE of Input i in the VOE-CD function corresponds to the cost share of input i:

$$\varphi_i = \frac{P_i^X X_i}{\sum_{j=1}^{n} P_j^X X_j}$$

The demand for inputs with a VOE-CD production function

- **Linear input demand:** the demand for each factor as a positive function of output and a negative function of the relative prices between production factors:

$$\dot{X}_i = \dot{Q} - \sum_{\substack{j=1 \\ j \neq i}}^{n} \eta_{ij} \varphi_j (\dot{P}_i^X - \dot{P}_j^X)$$

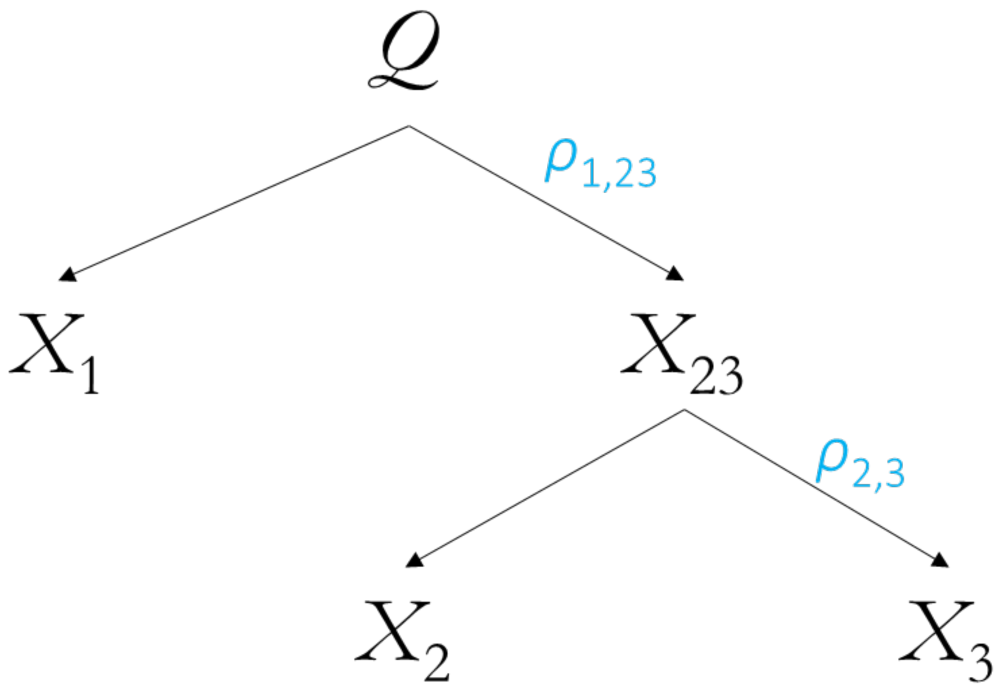
- Particular case: CES production function
  - A constant ES between inputs : $\eta_{ij} = \eta$

$$\dot{X}_i = \dot{Q} - \eta (\dot{P}_i^X - \dot{P}^Q)$$

$$\dot{P}^Q = \sum_{i=1}^{n} \varphi_i \dot{P}_i^X$$

Substitution properties of a nested CES system: the case of 3 inputs

- X_1 : Labor
- X_2 : Capital
- X_3 : Energy



Level 1

Level 2

$$\eta_{1,2} = \eta_{1,3} = \rho_{1,23}$$

Sign of the explicit ES between each pair of inputs is ambiguous → $\eta_{2,3} = \dfrac{\rho_{2,3} - \rho_{1,23}.\varphi_1}{1-\varphi_1}$

The panel construction

- We construct a panel that distinguishes the inputs expressed in volume and their price for 54 sectors and which covers 14 period of time and 42 countries.
  - The « WIOD 2016 release » provides World Input-Output and Socio-Economic Account tables from 2000 to 2014, both in current and previous year prices.

- We consider four inputs: Capital (K), Labor (L), Energy (E) Material (M) and the output Y
  - Energy and Materials variables are constructed from the World Input-Output Tables
  - Input Labor and Capital variables are constructed from the Socio-economic accounts tables

- Applying the chained-price methodology, we distinguish the economic volumes (k, l, e, m, y) to their prices $(p^K, p^L, p^E, p^M, p^Y)$ from which we compute the annual growth rates $(\dot{k}, \dot{l}, \dot{e}, \dot{m}, \dot{y}, \dot{p}^L, \dot{p}^E, \dot{p}^M, \dot{p}^Y)$

- We end up with 602 observations for each sectors (14 periods x 43 countries)

The econometric strategy

- We use a Seemingly Unrelated Regression Model (Zellner, 1961; Avery, 1977; Baghati, 1980) where the system of four equations is composed of the input demands K,L,E,M and estimate the elasticity of substitution η^{ij} parameters between the inputs i & j

- We introduce time and country fixed-effects ($\mu^{j}_{s,t}, \mu^{j}_{s,r}$)

- We test three specifications of the production function
  - A VOE-Cobb Douglas specification where there is no implicit nesting structures
  - A Nesting structure of the form (K(L(EM))
  - A Nesting structure of the form (K(E(LM))

- We impose cross-constraints restrictions on the elasticities such that $\eta^{ij} = \eta^{ji}$ in the VOE-Cobb Douglas case

The econometric strategy

- For the VOE-Cobb Douglas case, the system of equations is written:

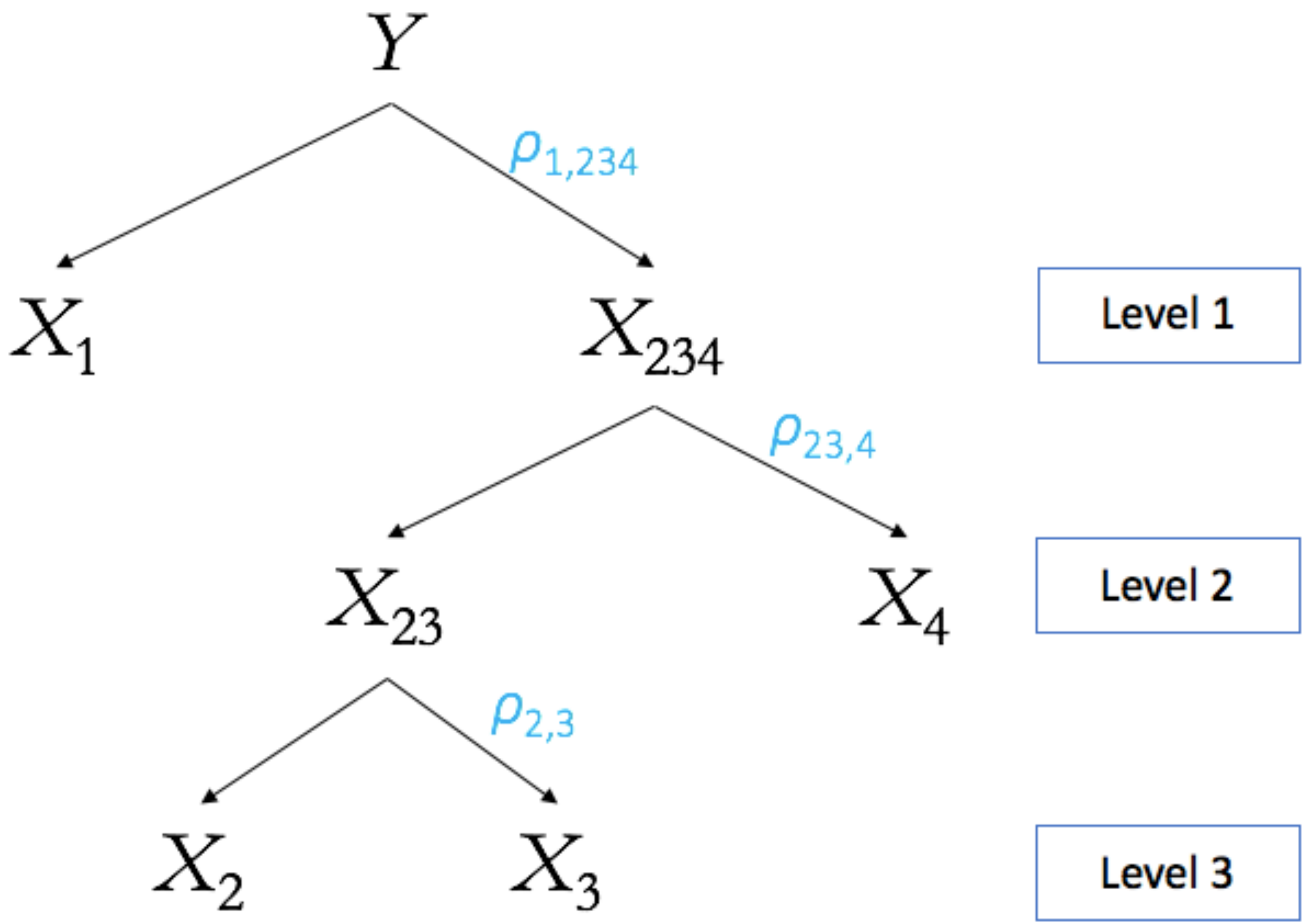
$$\dot{k}_{s,r,t} - \dot{y}_{s,r,t} = \alpha_s^k + \eta_s^{k,l} \varphi_{s,r,t-1}^l (\dot{p}_{s,r,t}^k - \dot{p}_{s,r,t}^l) + \eta_s^{k,e} \varphi_{s,r,t-1}^e (\dot{p}_{s,r,t}^k - \dot{p}_{s,r,t}^e) + \eta_s^{k,m} \varphi_{s,r,t-1}^m (\dot{p}_{s,r,t}^k - \dot{p}_{s,r,t}^m) + \mu_{s,r}^k + \mu_{s,t}^k + \varepsilon_{r,s,t}^k$$

$$\dot{l}_{s,r,t} - \dot{y}_{s,r,t} = \alpha_s^l + \eta_s^{l,k} \varphi_{s,r,t-1}^k (\dot{p}_{s,r,t}^l - \dot{p}_{s,r,t}^k) + \eta_s^{l,e} \varphi_{s,r,t-1}^e (\dot{p}_{s,r,t}^l - \dot{p}_{s,r,t}^e) + \eta_s^{l,m} \varphi_{s,r,t-1}^m (\dot{p}_{s,r,t}^l - \dot{p}_{s,r,t}^m) + \mu_{s,r}^l + \mu_{s,t}^l + \varepsilon_{r,s,t}^l$$

$$\dot{e}_{s,r,t} - \dot{y}_{s,r,t} = \alpha_s^e + \eta_s^{e,k} \varphi_{s,r,t-1}^l (\dot{p}_{s,r,t}^e - \dot{p}_{s,r,t}^k) + \eta_s^{e,l} \varphi_{s,r,t-1}^l (\dot{p}_{s,r,t}^e - \dot{p}_{s,r,t}^l) + \eta_s^{e,m} \varphi_{s,r,t-1}^m (\dot{p}_{s,r,t}^e - \dot{p}_{s,r,t}^m) + \mu_{s,r}^e + \mu_{s,t}^e + \varepsilon_{r,s,t}^e$$

$$\dot{m}_{s,r,t} - \dot{y}_{s,r,t} = \alpha_s^m + \eta_s^{m,k} \varphi_{s,r,t-1}^k (\dot{p}_{s,r,t}^m - \dot{p}_{s,r,t}^k) + \eta_s^{m,l} \varphi_{s,r,t-1}^l (\dot{p}_{s,r,t}^m - \dot{p}_{s,r,t}^l) + \eta_s^{m,e} \varphi_{s,r,t-1}^e (\dot{p}_{s,r,t}^m - \dot{p}_{s,r,t}^e) + \mu_{s,r}^m + \mu_{s,t}^m + \varepsilon_{r,s,t}^m$$

The Nesting structure



The explicit elasticities of substitution for the nested cases

- For the K(L(EM)) case, developping the nested system of equations and rearranging the terms to express the explicit demand for each input leads to the following relation between the elastictity of subsitution ρ and η :

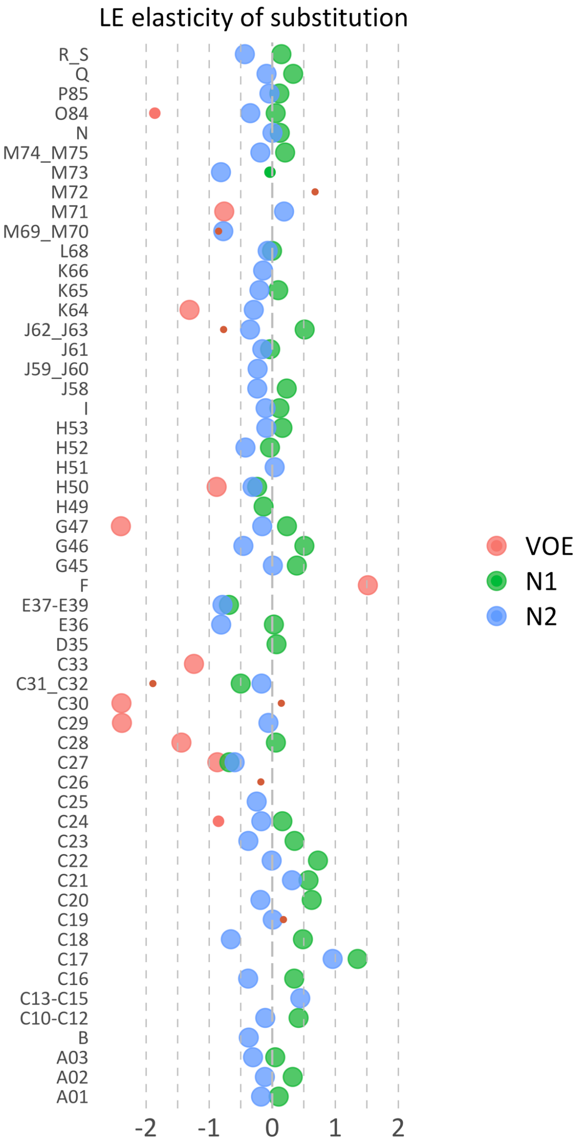
$$\eta^{k,l} = \eta^{k,e} = \eta^{k,m} = \rho^{k,lem}$$

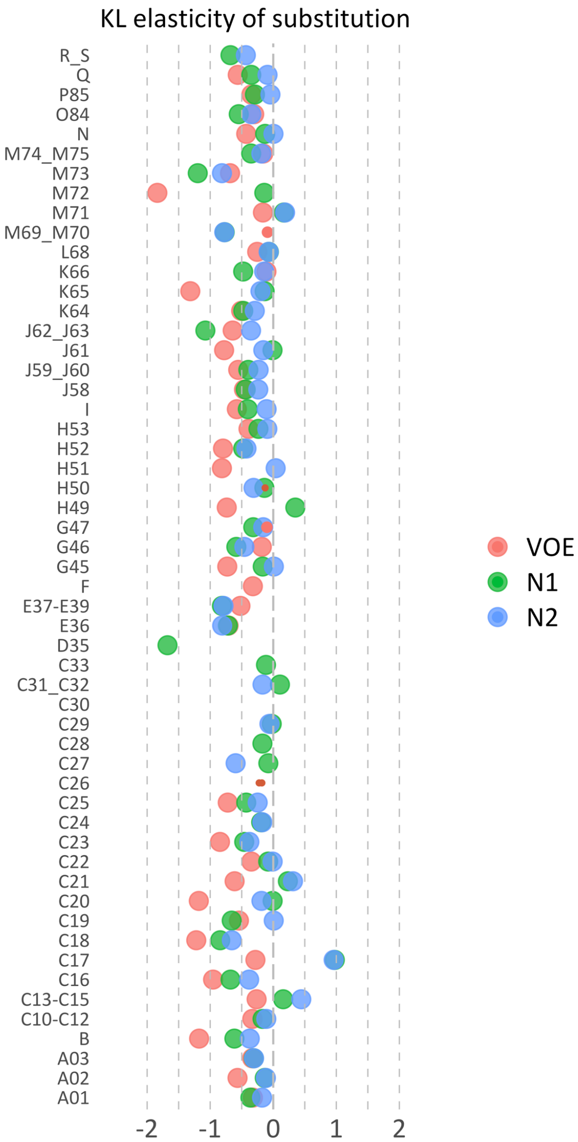
$$\eta^{l,e} = \frac{\rho^{l,e}}{1-\varphi^{k}-\varphi^{m}} - \frac{\rho^{k,lem}\varphi^{k}}{1-\varphi^{k}} - \frac{\rho^{k,lem}}{1-\varphi^{k}} - \frac{\rho^{em,l}\varphi^{l}}{(1-\varphi^{k})(1-\varphi^{k}-\varphi^{l})}$$

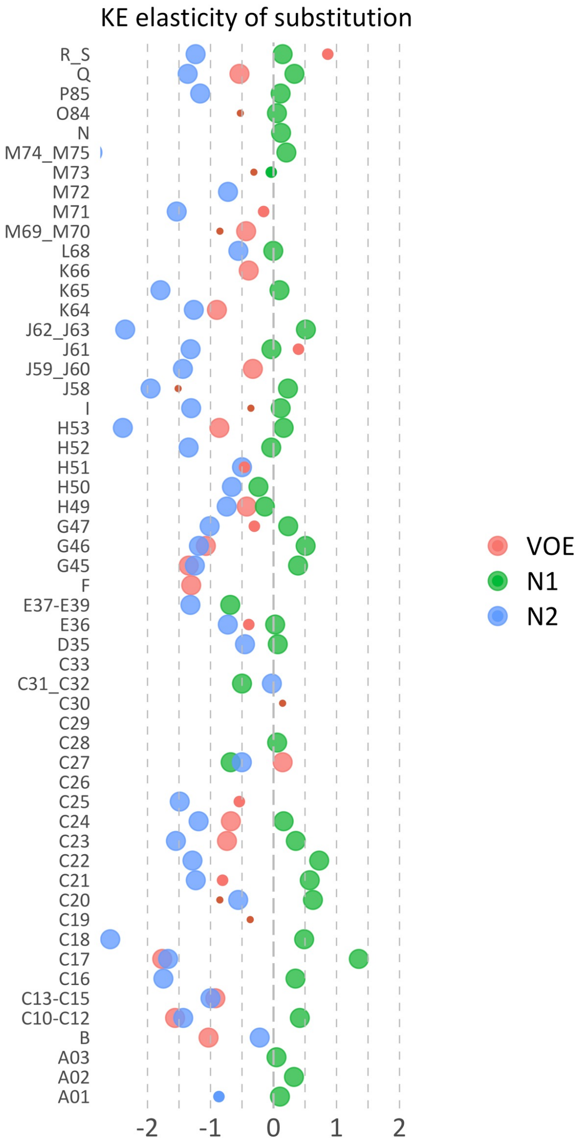
$$\eta^{l,m} = \eta^{m,e} = \frac{\rho^{em,l} - \rho^{k,lem}\varphi^{l}}{(1-\varphi^{l})}$$

- For the estimation N1 = K(L(E(M))) ; N2 = K(E(L(M)))

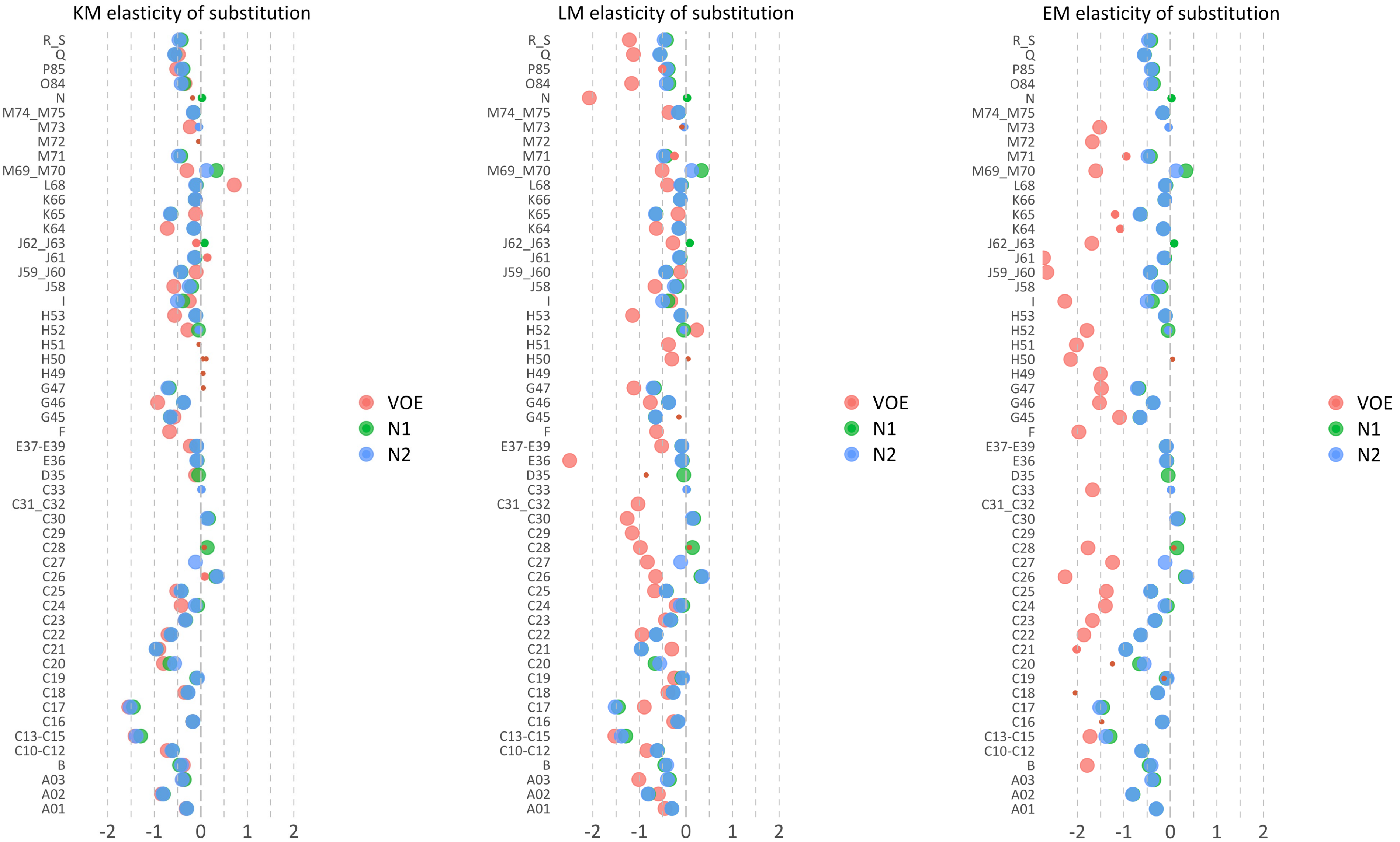
Estimation results







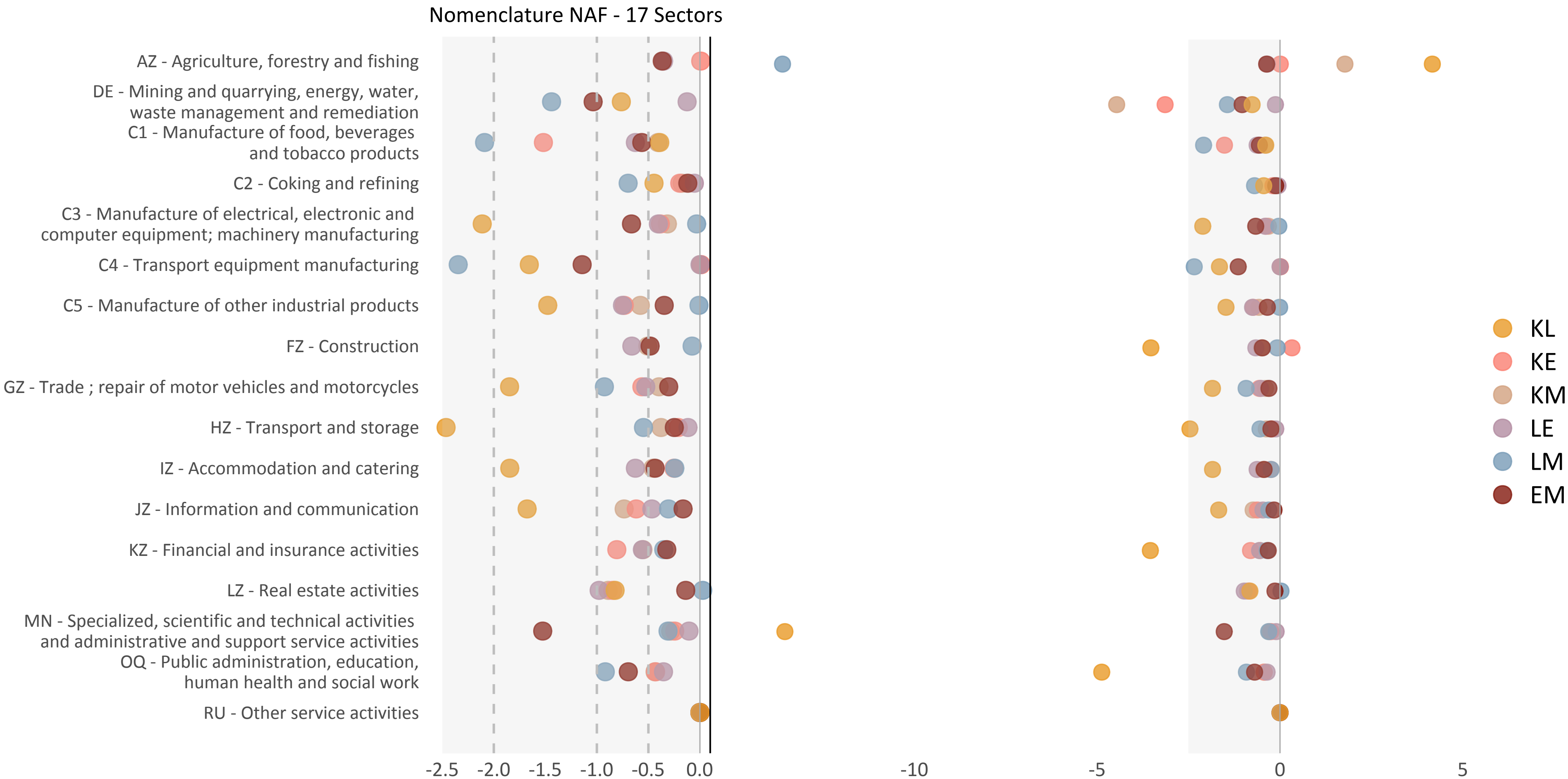
Estimation results



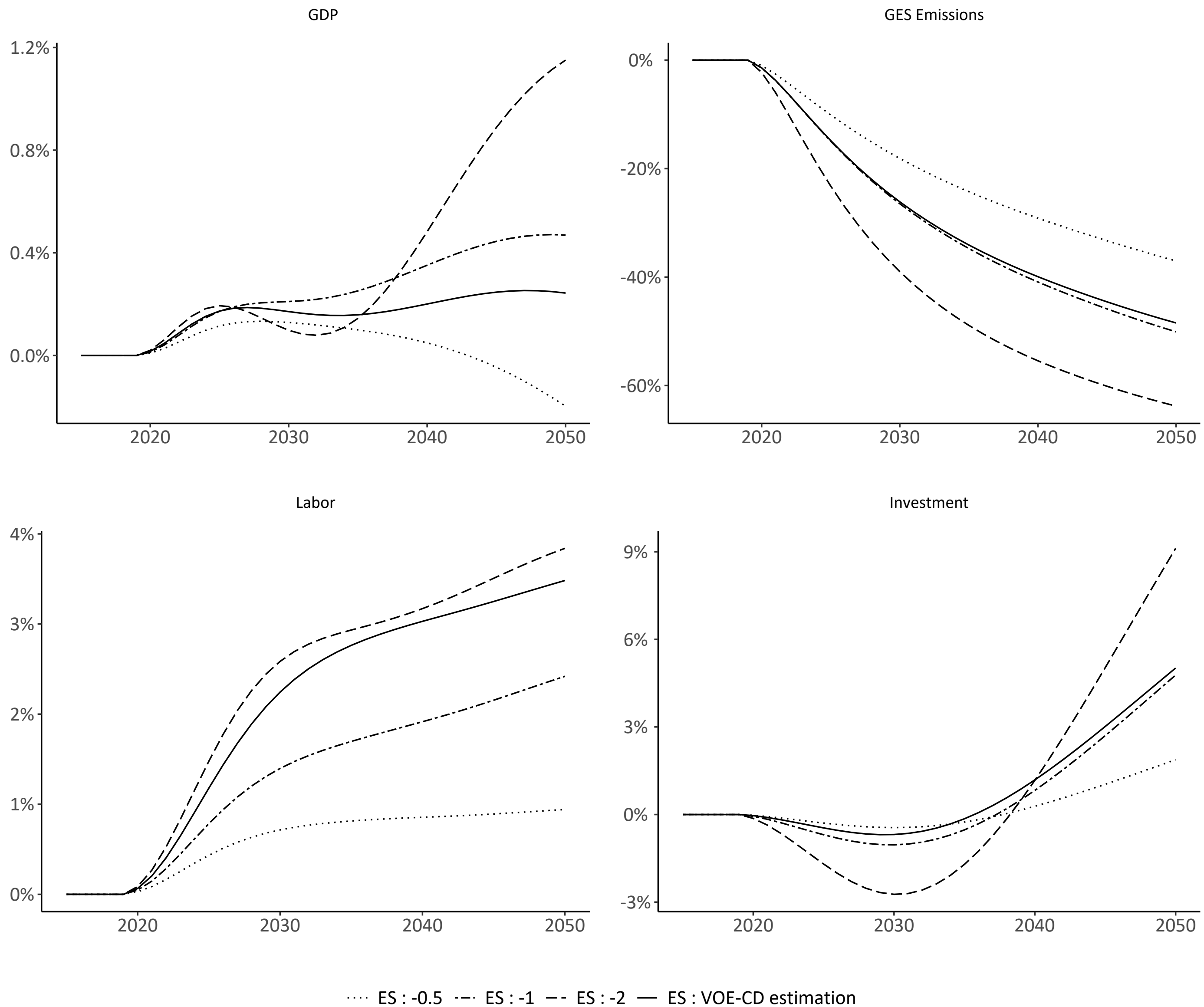
CGE simulations

- In order to establish the effect of the elasticities of substitution distribution on simulations results, we proceed to conduct sensitivity tests
- We consider three ad-hoc case + the results of the econometric estimation
  - ES = -0.5 ; ES = -1 ; ES = -2
- We take a 17 sectors segmentation version of the economy of the model ThreeME
- We apply a fully-recycled carbon tax shock (Quinet value trajectory)

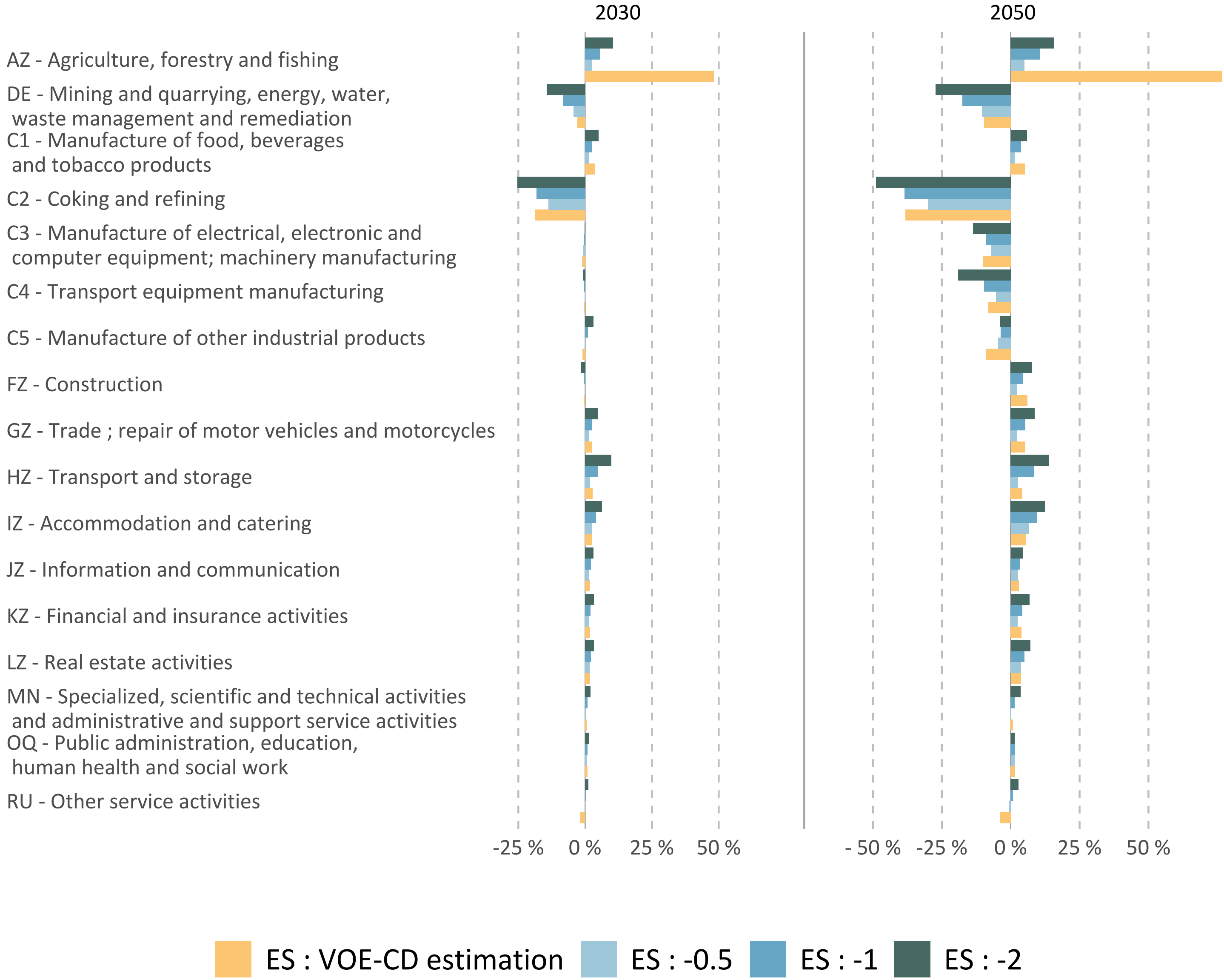
CGE simulations



CGE simulations: macroeconomic results



CGE simulations: Sectoral results



Discussion

- We consider a generic form of the production function which allows to relax the nesting hypothese on the inputs combination.
- We construct a panel dataset to proceed to econometric estimations of the production function, testing different specifications of nesting structures.
- We highlight the fact that the capital-energy relation seems to indicate substituability rather than complementarity, when using the VOE-Cobb Douglas specification
- From simulations conducted with the estimated ES, we can see for that the GDP impact is relatively lowered for a similar range of emissions reduction (ES: -1), and more job intensive
- Questions the sectoral effect at the macroscale though. Attention shall be carried on the dataset from which the panel is constructed (MRIO data transformation process which might be error prone)