An Information Theoretic Approach to Estimating Matrices of Economic Data

Scott McDonald and Sherman Robinson

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Address for correspondence:

Sherman Robinson

Emeritus Fellow, IFPRI, Washington DC

Email: S.Robinson@cgiar.org

1. Introduction

> *"[I]t is **impossible** to establish by direct estimation a system of national accounts free of statistical discrepancies, residual error, unidentified items, balancing errors and the like since the information available is in some degree incomplete, inconsistent and unreliable. Accordingly, the task of measurement is not finished when the initial estimates have been made and remains incomplete until final estimates had been obtained which satisfy the constraints that hold between their true values." (Stone, 1982, p 186, emphasis added).*

All quantitative analyses of national economic systems, micro or macro focused, are based on estimates of national accounts. National accounts data are usually presented in double-entry receipt/expenditure tables. They can also be presented as single-entry accounts in a matrix format, of which the most comprehensive (square) form is a Social Accounting Matrix (SAM).[1] All national accounts data, in whatever framework, are point estimates subject to sampling and measurement errors because, in large part, they are the products of underdetermined estimation problems. The discussion in this paper focuses on the estimation of SAMs, although the techniques and arguments developed are applicable to the estimation of a wide range of data matrices, e.g., Supply and Use tables (SUT), bilateral trade balances, demographic accounts, etc. The paper reviews different approaches to estimating SAMs and argues that recent methods based on a combination of Shannon information theory and Bayesian statistics are preferable. They have a strong theoretical foundation, incorporate prior information (statistical and theoretical) in a flexible manner, and are feasible to implement empirically.

Models based on SAMs assume that the data accurately reflect the operation of the economy. Erik Thorbecke emphasized the importance of getting the data right.

> "The issue of whether the SAM is deterministic or stochastic is crucial as the SAM provides the underlying data set upon which simple SAM-multiplier analyses and more complex Computable General Equilibrium Models (CGEs) are calibrated. Increasingly, these models are used to explore and simulate the impact of policies and exogenous shocks on the whole socio-economic system. An erroneous or inaccurate SAM invalidates the results obtained from these models" (Thorbecke, 2003, p 186)

It is a theoretical requirement that a SAM, and other economic data matrices, are 'complete' and 'consistent': 'complete' in that all transactions/values are recorded and 'consistent' in that each expenditure value is matched correctly with an income value. For any SAM that is both complete and consistent, row and column totals will equate.

The early compilers of SAMs adopted strategies that involved confronting data from different sources with each other; taking a subjective view on the reliability of the different sources and then attempting to satisfy the SAM accounting constraints of row/column sum equality (see Pyatt *et al*., 1977; Pyatt, 1991). In this context 'balancing' a SAM involved reconciling small residual differences. Stone (1977, p. xxi) responded to this laudable but "labourious method" by asking whether "still better results could not be obtained by applying a formal, mathematical treatment rather than *ad hoc* manipulations to our subjective assessment of

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1 The data for all whole economy models, whether founded on macro or microeconomic principles, can be presented in SAM format, which is the foundation of the UN-SNA System of National Accounts (Pyatt, 1987).

reliability" (p xxi). Since Stone was intimately familiar with existing mathematical methods then used to reconcile matrices of data, it is reasonable to conclude that Stone's challenge was to find empirical methods that would support estimating rather than simply reconciling/balancing data matrices.

Thorbecke (2003) moved the debate about estimation strategy beyond the development of mathematical techniques by arguing that it is not enough for the techniques to provide a 'mathematical/statistical' solution, but rather they must also incorporate recognition that each cell in a SAM is "an estimate arrived at on the basis of data containing sampling and measurement errors" (Thorbecke, 2003, p 185); estimation strategies must be grounded in theoretically consistent processes.

In the literature on SAM estimation there are many references to "information theory" as providing a framework for dealing with data from many sources to address both the Stone and Thorbecke challenges. There are two broad approaches, both of which use metrics of entropy and cross-entropy: $x \cdot log(x)$ and $x \cdot log\left(\frac{x}{x^0}\right)$ where x is a variable of interest and x^0 is a prior estimate of x. In the first approach, the entropy and cross-entropy metrics are assumed to reflect a broad descriptive concept of "information" and are applied to positive variables measured in various units such as values, coefficients, or shares. The second approach is based on information theory as developed by Claude Shannon (1948) where x and x^0 are probabilities. Zellner (1988) describes the close links between Shannon entropy and Bayesian statistics, supporting an approach to estimation using efficient "information processing rules" that use all available information and do not assume any information that is not available. Both approaches have been used in SAM estimation and are discussed in this paper.

The Shannon-entropy method for estimating a SAM is described in detail and addresses both the Stone and Thorbecke challenges. Widely used RAS-based estimation methods (biproportional row/column adjustments) can be shown to use the first descriptive entropy approach to estimation. They and other similar loss minimisation (mathematical) methods (see Günlük-Şenesen and Bates, 1988), achieve the objective of deriving a consistent SAM, but they satisfy neither Stone's nor Thorbecke's challenges. The Stone-Byron method[2] is an advance on RAS-based methods in that subjective judgements enter "at a second-order rather than first-order level" (Round, p 177, 2003).

The remainder of this paper is structured as follows. Section 2 provides a brief overview of the structure of a SAM and the development of RAS and 'traditional' mathematical estimation methods. The third section recasts the estimation problem as a Shannon information theoretic problem that uses maximum and cross entropy probability metrics as the mathematical optimisation method. It also demonstrates how the information theory approach provides an effective framework that incorporates a comprehensive range of information. The last section provides concluding comments.

2. Social Accounting Matrices (SAM)

A SAM provides information about transactions between agents in an efficient and, ultimately, simple way, and in a manner that is consistent with the aggregate/macro accounts for the system of national accounts (SNA). The development of the first fully articulated SAMs was largely undertaken by the Cambridge Growth Project (Cambridge, Department of

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2 See Stone (1977); Byron (1978), van der Ploeg (1982) and, for a summary, Round (2003).

Applied Economics, 1962-74), while the first SAMs for developing countries were developed by Graham Pyatt and associates (Pyatt *et al*., 1977; Pyatt and Round, 1985).

2.1. SAM Principles

The guiding principles of a SAM are the concept of the circular flow and the requirements of -single-entry bookkeeping. A SAM captures the full circular flow in a square matrix, **T**, where the cell entries, t_{ij}, are transaction values (TV), with the row entries recording the incomes received and the column entries recording the expenditures by the accounts. Each cell, t_{ij}, simultaneously records the income to account *i* from account *j* and the expenditure by account *j* to account *i*. Therefore, the total expenditures by each account must be exactly equal to the total receipts for each account: hence the respective row and column sums for a SAM must equate, i.e.,

$$y_j = \sum_i t_{i,j} = \sum_i pr_{i,j} * q_{i,j} = \sum_j pr_{i,j} * q_{i,j} = \sum_j t_{i,j} = y_i \qquad (1)$$

where y_j and y_i are the column and row sums and each TV, $t_{,jj}$, can be expressed as the product of prices, $pr_{i,j}$, and quantities, $q_{i,j}$. The elements ($a_{i,j}$) of the matrix, **A**, of column coefficients is then defined as

$$a_{i,j} = \frac{t_{i,j}}{y_j} \qquad (2)$$

such that the column sums for the matrix **A** are all equal to one.

The requirements of single-entry bookkeeping ensure that if the national accounts are fully reconciled, the corresponding SAM would be complete and consistent. A SAM can be consistent, i.e., row and column totals equate, but incomplete because for some cells the entries are missing or are partial records, i.e., some recorded transactions must have been distorted to render the SAM consistent.

SAMs are generally constructed with 6 types of accounts, where each type may contain multiple sub accounts: (1) commodities/products, (2) activities/industries, (3) factors, (4) institutions, (5) saving/investment, and (6) rest of the world (see Table 1).[3] The commodity accounts record supply and demand, both domestic and exports, for different commodities that have been sourced domestically or imported and details about commodity taxes and trade and transport margins. The activity accounts record domestic production using intermediate inputs and primary factors plus any taxes paid by activities. The institutions receive incomes from the sales of factors to domestic activities and the rest of the world, from tax revenues, inter-institutional transfers, and transfers from the rest of the world, e.g., migrant remittances. The savings account records savings by domestic institutions, depreciation and the (net) current account balance; the savings are used to fund investment. The rest of the world accounts record all transactions between the host economy and the rest of the world.

The format of the inter-industry components of a SAM in Table 1 is implicitly that of Supply and Use Tables (SUT). This means that the sub matrix activities/commodities, i.e., domestic production/supply, is not diagonal, and indeed may not be square, in terms of the collected

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3 The decision to use a structure that is more reflective of the 1968 SNA is not a comment on the 1993 SNA, rather it is an expositional device chosen simply because it allows the use of a more parsimonious table.

data.[4] This is arguably consistent, in terms of the SNA, where Input-Output Tables (IOT) are derived from Supply and Use Tables (SUT) by linear transformations. While the discussion in this paper assumes that data will be collected for the SAM in SUT format, and the estimated SAM will use a SUT format, the estimation methods explored could use IOT based SAMs.[5]

An important property of a SAM is the 'Law of One Price', whereby all transactions in a row have the same price. From the accounting identities it follows that the average prices for transactions are equal to the average costs that are defined by the entries in the respective columns (see Pyatt, 1988), i.e.,

$$y_j = \sum_i pr_i * q_{i,j} = \sum_j pr_i * q_{i,j} = y_i \tag{3}$$

where there is single price in each row, which means the quantities in each row are homogenous. Hence, the information content of the column entries is critical for economic models, especially if they are price driven, because they define how prices in economic systems are determined.[6]

4 National accounts typically use a principal commodity/product schema to allocate production establishments to activities, i.e., the commodity account labels define the activity account labels and then the activities/commodities sub matrix will be square but not diagonal.

5 This does not imply that either format is preferable for use when calibrating economic models. The decision is solely based on the practicality of estimating a SAM.

6 If an homogenous commodity is 'sold' at different prices there must be inseparabilities, e.g., commodities that can be sold on markets, incurring taxes and trade and transport margins, or home consumed. Such information should be reflected in the account structure of a SAM.

Table 1 A Stylized Social Accounting Matrix

	Commodities	Activities	Factors	Households	Enterprises	Government	Capital	RoW	Total
Commodities	0	Intermediate Inputs	0	Household Demand	Enterprise Demand	Government Demand	Investment Demand	Commodity Exports	Total Commodity Demand
Activities	Domestic Production	0	0	0	0	0	0	0	Gross Output
Factors	0	Factor Demand	0	0	0	0	0	Factor Income from RoW	Factor Income
Households	0	0	Distributed Factor Income	Inter-Household Transfers	Distributed Dividends	Transfers	0	Remittances	Total Household Income
Enterprises	0	0	(Un)Distributed Factor Income	0	0	Fixed (Real) Transfers	0	Transfers	Total Enterprise Income
Government	Tariff Revenue VAT Other Taxes on Commodities	Indirect Taxes on Activities	Distributed Factor Income	Direct Taxes on Household Income	Direct Taxes on Enterprise Income	0	0	Transfers	Total Government Income
Capital	0	0	Depreciation	Household Savings	Enterprise Savings	Government Savings (Internal Balance)	0	Current Account ‘Deficit’	Total Savings
Rest of World	Commodity Imports	0	Distributed Factor Income	Remittances	Transfers	Transfers	0	0	Total ‘Expenditure’ Abroad
Total	Total Commodity Supply	Total Activity Inputs	Total Factor Expenditure	Total Household Expenditure	Total Enterprise Expenditure	Total Government Expenditure	Total Investment	Total Income from Abroad	

2.2. Developing a SAM

An appropriate method for developing a SAM starts with the theory of social accounting and the primary data, e.g., surveys of activities,[7] households, labour force, trade accounts, government tax accounts etc., and uses these data sources to develop a prior estimate of the complete SAM. Working within an accounting structure derived from theory, these data need to be evaluated for (i) coverage of the data sources, (ii) quality of the primary data, and (iii) replicability of the methods used to reconcile differences between data from different primary sources, e.g., tobacco consumption from household surveys and supply from production and trade data, so that new data sources and/or re-evaluations of data quality can be incorporated in an objective way.

Stone's labourious process of developing best point estimates for a prior SAM of transactions is critical. As parts of the process, it is important to identify transactions that the accounting structure and economic theory dictate have zero values, e.g., sub matrices of the SAM in Table 1 with a zero entries. Similarly, other transactions such as between an agent and itself may also be zero. Theory and/or empirical evidence may indicate that certain TVs should be non-zero, but data may be missing, indicating an incomplete SAM. When evaluating the primary data, it is also important to form judgements about the reliability of the point estimates. Such judgements may be statistical, e.g., variances derived from household survey data, or subjective, e.g., informed judgements about error bounds without information about the error distributions.

Since "the information available is in some degree incomplete, inconsistent and unreliable" (Stone, 1982, p 186), the prior SAM will likely be inconsistent and may be incomplete. This means that some estimation method, subjective and/or mathematical/statistical, is a *sine qua non* for the estimation of a complete and consistent SAM.

As part of the data compilation process, analysts could compile a series of aggregate totals that may be measured with or without error, which can be specified as control totals in estimation. These may include estimates of row and column totals, individual TVs, tax revenues, total value of intermediate inputs by activities, macro aggregates such as GDP and components.

2.3. Other Estimation Methods: A Brief Review

For any SAM, **T**, there are an infinite number of matrices that could satisfy the consistency condition that the row and column totals equate. All known SAM estimation methods use some form of bi-proportional or information-loss minimisation methods to select between the possible solution matrices. These optimands can be expressed as some form of weighted summation of the differences between the solution values and prior estimated values, with the solution and prior values expressed in natural units (values or coefficients) or logarithms, depending on the method. For all practical purposes the values can be expressed as coefficients or transaction values (TVs); below it will be assumed that the cells, t_{ij}, of the matrix **T** are TVs unless otherwise stated.

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7 Activities, and institutions, report (in surveys or censuses) their expenditures on commodities and factors, while trade data are collected for commodities, but activities report their supply in terms of multiple commodities. Hence a preference for the SUT format.

The three classes of methods considered below demonstrate the importance of the estimates of the prior TVs, t_{ij}^{0}, and therefore the critical importance of the labourious method whereby data from different sources are fully evaluated when deriving prior estimates of TVs. It also transpires that most of the estimation issues of concern remain those faced in the 1950s and 1960s, despite these advances in computational power and solution algorithms that have greatly expanded the domain of practical estimation methods.

If the prior TVs, t_{ij}^{0}, are very close to the solution TVs, t_{ij}, then the choice of minimand may be of little importance where the estimation problem is to reconcile minimal differences in the control totals to ensure consistency. This process can be labelled ‘balancing’ since the estimation process has been largely completed when developing the prior.

2.3.1. RAS Methods

The minimand for the RAS (bi-proportional) method (Bacharach, 1965 and 1970) can be written as a descriptive cross-entropy (CE) function in coefficient or value:

$$\sum_{i,j} a_{ij}\, ln\left(\frac{a_{ij}}{a_{ij}^{0}}\right) \text{ or } \sum_{i,j} t_{ij} ln\left(\frac{t_{ij}}{t_{ij}^{0}}\right) \tag{4}$$

where a_{ij} are the solution column coefficients and a_{ij}^{0} are the prior estimates of the column coefficients, subject to the row and column totals equating. Note that the weights for the logarithmic ratios are the solution coefficients or values. Overviews of the RAS method and variants are provided by Lecomber (1975), Allen (1974), Allen and Lecomber (1975), and Miller and Blair (2009, pp 313-342). The last of these and includes references to the continuing use of RAS methods.

In the literature, the use of the entropy and cross-entropy (CE) metrics in this approach was viewed as reflecting information theory, but not in terms of probability distributions that was the focus of Shannon’s work. For example, Theil (1967) discusses the information content of an input-output matrix and Bacharach (1973) has a similar discussion, citing Theil (1967). [8] Note that this wider view of information theory, with the use of entropy metrics, appears in many fields, not just economics, e.g,. see Chen *et al*. (2021), Golan (2018), Kapur (1989), and Kapur and Kesavan (1992).[9]

An early approach to solving the first-order conditions of RAS CE minimization with known row/column sums was to iterate proportional adjustment on rows and columns until successive row and column adjustments converged on equality. That this bi-proportional adjustment method was equivalent to solving the first-order conditions was known at the

8 Uribe *et al*., (1965), as reported by Theil (1967, pp 357-65), drew on this approach to information theory to estimate international trade flows between regions where total exports and imports are known for each region. See also Georgescu-Roegen (1971)

9 Kapur and Kesavan (1992) and Golan (2018) have extensive discussions of this broader view of information theory and links to Shannon entropy and cross-entropy measures.

time.[10] This simple computational method was widely used at a time when computer algorithms for solving nonlinear programming problems were in their infancy.

The substantive issues confronted by users of RAS methods (originally) in the 1960s and 1970s all derived from circumstances in which a RAS method was used to do more than reconcile very small differences between known and certain row and column totals, i.e., reconcile/balance a matrix. When using mathematical methods to 'update' a data matrix using a prior based on the data from a matrix for a previous period, e.g., using data from a 1954 IOT as priors to estimate a 1960 IOT (Stone *et al.*, 1963), the results were unreliable (Allen, 1974; and Lynch, 1979). It is evident that a root cause of the unreliability derives from the failure of the priors to provide accurate representations of structural changes in economic systems between the prior and solution matrix.

It was recognised, early, that "if the accuracy of RAS and similar procedures is to be improved more information must somehow be incorporated, information on the moments of individual elements and on the putative accuracy of row and column totals" (Lecomber, 1975, p 9). Variants of the RAS routines were developed to improve the 'perceived' performance of the RAS metric; these included adding information about specific cells,[11] the measurement of row and column control totals and selected aggregates with error [12].

Early reviews of the RAS method and its variants (Lecomber, 1975; Allen and Lecomber, 1975) concluded that minimum information variants were "reliable only under favourable circumstances" and that "[P]rojections using RAS updates are extremely unreliable" with "unacceptable upward bias" (Lecomber, 1975, p 21). The incorporation of additional information can markedly improve on the performance of RAS variants if "extraneous information is treated …. as accurate" but if "allowance is made for its lack of reliability, the incorporation of such information leads to a very substantial improvements over RAS" (Allen and Lecomber, 1975, p 51).

The conclusion that "as far as possible all available information should be used, even that which is not fully reliable or appropriate and that, where possible, a subjective assessment of the reliability of information should be taken into account in a more or less formal way in the estimation procedure" (Lecomber, 1975, p 22) remains valid. This understanding appears to have been increasingly replaced by an emphasis on the properties, especially speed, of the mathematical/statistical method as, at least, a partial replacement for primary data/information.

Golan, Judge, and Robinson (1994) and Robinson, Cattaneo, and El-Said (2001) used the cross-entropy (CE) minimand in terms of column coefficients of a SAM and treated them as analogous to probabilities since they sum to one. In effect, they used a descriptive CE metric but discuss the flexibility of the approach to incorporate many kinds of additional information

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10 The minimand for the RAS method was known before July 1964, the date Bacharach's (1965) paper was submitted.

11 Standard RAS method can be expressed as $\mathbf{X} = \hat{\mathbf{r}}\mathbf{X}^0\hat{\mathbf{s}}$, where $\mathbf{X}$ is the solution matrix, $\mathbf{X}^0$ the prior estimates and $\mathbf{r}$ and $\mathbf{s}$ the row and column multipliers, subject to the row and column totals equating. A modified RAS method can be expressed as $\mathbf{X} = \mathbf{K} + \hat{\mathbf{r}}\left(\mathbf{X}^0 - \mathbf{K}\right)\hat{\mathbf{s}}$, where $\mathbf{K}$ is a matrix of known values.

12 Selected aggregates can be expressed as row/column control totals, with or without error, by reorganisations of the data matrix.

as constraints in the estimation procedure.[13] The use of column coefficients as proxies for probabilities encounters a problem with negative column coefficients, as also experienced with RAS; this was countered by transposing negative TVs and adjusting the row and column control totals before computing the column coefficients.[14]

Given that the cross-entropy (CE) function in terms of column coefficients is consistent with the objective function of the RAS method, (McDougall, 1999) argues that CE and RAS are ‘friends’ but this sidesteps the differences in the solution methods where the CE approach estimates all the coefficients independently whereas RAS imposes an assumption of bi-proportionality.

2.3.2. Information Loss Methods

Similarly, the loss minimisation methods can be summarised as

$$\sum_{i,j} \left| t_{ij} - t_{ij}^{0} \right|^{\frac{z}{v}} \qquad (5)$$

where *z* may take any value and *v* are variances.[15] Both are minimised subject to, at least, the constraints that row and column totals equate. The best known of these is the Stone-Byron method, which assumes that each element of the prior matrix has been estimated with error.

This approach to errors was used by Stone *et al.*, (1942) and Stone (1977) and operationalised as the Stone-Byron method, following Byron (1978), as a generalised least squares loss minimisation method. This method has been refined (see van der Ploeg, 1982) and subjected to testing (see Byron, 1996). Under assumptions that the error terms are normal this approach satisfies maximum likelihood criterion (see Stone *et al*., 1942) and was operationalised by Weale (1985).

The Stone-Byron method requires information for each cell in the matrix to derive estimates of the variances and the appropriate error distributions. These data are unavailable for an undetermined problem, and therefore these methods require strong ‘subjective’ judgements. These subjective judgements enter “at a second-order rather than first-order level” (Round, 2003, p 177), but it fails to satisfy Stone’s stated objective because, as Stone stated (1977, p xxii), the required variance matrix “can only be based on subjective impressions of the investigator”. The method addresses many of the concerns with the RAS method but is limited by the functional forms of the error terms and does not provide a straightforward way of incorporating information on sub-aggregate totals.[16]

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[13] In their simple example used for illustration, Golan, Judge, and Robinson (1994) assume knowledge of row/column sums, so their solution is equivalent to the RAS method. Robinson, Cattaneo, and El-Said (2001) impose a variety of informative constraints on the estimation.

[14] Günlük-Şenesen and Bates (1988, pp 475-6) observed that negative transactions could be problematic when deriving a solution, but that a corrected iterative solution method can be devised.

[15] See Günlük-Şenesen and Bates (1988, p 476).

[16] Some sub-aggregate totals can be incorporated by an appropriate organisation of the SAM.

2.3.3. Weighted Difference Methods

A common optimand is the sum of the weighted squared differences between the prior and solution TVs where the weights are the prior TV (e.g., Friedlander, 1961)[17], i.e.,

$$\sum_{i,j}\left[\frac{\left(t_{ij}-t_{ij}^{0}\right)^{2}}{t_{ij}^{0}}\right] \tag{6}$$

subject to the constraints that the row and column 'control' totals are known and equate.[18] For an appropriate choice of weights (discussed below), this function is an approximation of the cross-entropy metric underlying the RAS approach and is similarly tractable to solve using modern computer algorithms.[19] However, it only uses information from the prior SAM and control row/column totals.[20] This limited use of information highlights the critical importance of the accuracy of the prior TVs, with respect to both the differences and the weights. These methods are likely to be useful only for the final balancing of a very accurate prior SAM.

One feature of these methods is that they are not sign preserving. This is an important issue when considering price formation in the presence of taxes and subsidies where the distinction between them is critical and was a deemed a major advantage of RAS methods.

2.3.4. A Least Squares Approximation to the CE metric

Golan *et al*., (1996, Proposition 3.3.1, pp 30-31) demonstrate that the CE objective function can be approximated by a weighted sum of squared differences:

$$CE=\sum_{k}p_{k}.\log\frac{p_{k}}{p_{k}^{0}}\approx\sum_{k}\frac{1}{p_{k}^{0}}\left(p_{k}-p_{k}^{0}\right)^{2} \tag{7}$$

for $p_{k}^{0}>0$, where p_k are posterior probabilities and p_k^0 are prior probabilities.

The weights in the exact CE objective function are the posterior probabilities, those in the approximation are the prior probabilities. In addition, the quality of the approximation depends on $p_{k}\approx p_{k}^{0}$ such that $\ln p_{k}/p_{k}^{0}\approx\left[\left(p_{k}/p_{k}^{0}\right)-1\right]$, i.e., the prior and posterior probabilities are close.

This function can be written in terms of SAM transaction values rather than coefficients by substituting the definitions of the column coefficients in terms of the transactions value. The result is a weighted sum of squared differences between solution and prior transactions, $t_{ij}\ and\ t_{ij}^0$. Define T_j^0 as prior normalized column sums so that they match the target column sums. In effect, the SAM is normalized so that it is adjusted by the first iteration of the RAS

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17 Matuszewski, *et al*., (1963) proposed a variant using a base weight sum of the absolute differences.

18 Friedlander's function relationship can be written as $\mathbf{X}=\hat{\mathbf{r}}\mathbf{X}^{0}+\mathbf{X}^{0}+\mathbf{X}^{0}\hat{\mathbf{s}}$.

19 Golan, Judge, and Miller (1996, p. 31) discuss the least squares approximation of the cross-entropy metric.

20 Appropriate organisations of SAM accounts can provide row and column totals that encompass aggregates.

biproportional adjustment method. Column sums hit targets and row sums do match. In this case, the weighted squared value difference approximation to the CE function is given by:

$$CE(A, A^0) \approx \sum_j T_j^0 \sum_i \frac{1}{t_{ij}^0} \left(t_{ij} - t_{ij}^0 \right)^2 \tag{8}$$

This weighted least-squares optimand differs from other least-squares functions used in the literature. Golan et al. (1996, pp 12-14) present an example to demonstrate that for underdetermined estimation problems such as SAM estimation, an unweighted least squares solution is likely to yield nonsensical results. Any least-squares function (weighted or not) of SAM values other than the CE approximation is suspect, at the very least introducing unwarranted information. With modern solution algorithms, minimizing the CE function is feasible and there is no reason to use any least-squares approach.

3. SAM Estimation with Shannon Entropy

Shannon information theory provides a "logical way for converting underdetermined problems into ill-posed ones" (Golan 2018, p 18) where the solution is in terms of a probability distribution. Adopting the 'principle of insufficient reason'[21], the estimation principles can be defined as:

1. use all the information available, and
2. do not use, or make assumptions about, information that are not available.

These principles are remarkably similar to the those that emerged from evaluations of the RAS methods in the 1960s and 1970s.

In this approach, it is not appropriate to make assumptions about either the error generating process or error distribution, e.g., bi-proportionality for the RAS or the variance matrix for the Stone-Byron methods, unless there are relevant supporting data. In essence the cells of a SAM are unknown parameters whose values must be estimated from observed data. The process of compiling a SAM is an underdetermined estimation problem since there are typically more cells/parameters to estimate than available data points, which means conventional statistical/econometric methods are not strictly appropriate for the derivation of inferences about the unknown parameters.[22] Shannon information theory provides a theoretically consistent framework within which parameters can be estimated when data are scarce and/or incomplete, defining the entropy metric as the unique appropriate choice of optimand.[23] This

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[21] The principle is commonly attributed to Bernoulli and Laplace (see Golan, 2018, p 4, Box 1.1).

[22] The Stone-Byron method renders the problem susceptible to conventional econometric methods – generalised least squares – by the imposition of (subjective) variances; hence the method side steps the problem of degrees of freedom.

[23] It is tempting to argue that the issues of scarce and incomplete information are particularly problematic for developing countries where data are particularly scarce and unreliable – measured with a lot of error. This appears to be true in relation to activities within the production boundary but outside of formal market transactions and where the values of transactions must be imputed, e.g., home production for home consumption in semi-subsistence economies, but this requires a presumption that data for developed economies are necessarily more complete, which may not be the case, e.g., the withholding/suppression of data for highly concentrated industries/activities, which is often the case for the food, especially sugar, industries in the EU. Moreover, this ignores the issues of inconsistent data, which may be more important than incomplete data, and there appears to be no absolute reason to believe this issue is related to the stage of development.

approach is consistent with Zellner's (1988) 'efficient information processing rule' and has strong links with Bayesian estimation.

The Shannon entropy approach exemplifies why the terms 'updating' and 'balancing' are arguably inappropriate in the context of mathematical methods used to generate new SAMs. 'Updating' has often referred to the derivation of a SAM for a later period primarily based upon new estimates of the total incomes/expenditures for accounts and the previous transactions data; in which cases the prior is assumed to provide a good representation of a future economic structure. Experience since the 1960s has demonstrated that this is not a reliable approach without substantial amounts of additional data. 'Balancing' has typically referred to the removal of inconsistencies given that calculated row and column totals differ from exogenously known totals. These methods may produce consistent SAMs, but the resultant SAMs are unlikely to be reliable and may be biased due an incomplete prior.

Ultimately the SAM estimation problem can be regarded as constituting two related sub problems; the assembling of prior information, i.e., what information should be provided, and the choice of objective function. At the same time, it is critical to ensure that the solution to the accounting problem does not undermine the economic content of the system, i.e., the information about price formation in the columns. The concern in this section is with the formulation of an objective function based on Shannon information theory and its implementation.

3.1. Choice of Objective Function

Arguably the prior information requirements are independent of the choice of objective function. However, as studies about the estimation of data matrices make clear, the choice of objective function is not straightforward since different objective functions require different additional information or assumptions. In this context it is worth quoting at length from Stone *et al*., (1942, p 112)

> "If it could be assumed that we knew how to measure the standard errors of our … estimates the answer would be simple; the weights we should use would be the reciprocals of the squares of these standard errors. The solution so obtained would give the minimum possible value of the sum of the squares of the standardised errors. Also, in cases where the error distributions are normal this solution would satisfy the criterion of maximum likelihood. In general, however, it will not be possible in what follows to measure the standard errors in the ordinary way and we shall have to resort to setting margins of error, …….. , such that they are the best estimates we can make of parameters analogous to standard errors. Accordingly, they must be thought of not in terms of a particular method of measurement but as limits which, however calculated, aim at providing information as to the precision of the estimates and as being so chosen that the chances are believed to be roughly two to one that truth lies within them. Each may be thought of as the standard deviation of the hypothetical distribution[24] of all possible independent estimates of an element. The errors which give rise to this distribution may be either logical or statistical in character. The first source of error is important since we are concerned with the probability that our estimate attains the true value of what we should have

[24] It is noted in a footnote of the original text: "In cases where this distribution is markedly skew or bimodal, the above method of estimating the standard deviation will be especially rough." (Stone, *et al*., 1942, p 112, footnote 1).

> measured had our definitions been faultless. In practice they may not be, so that our tables would not balance even if our measurements were perfect."

It is notable how much of the subsequent literature on reconciling national accounts data has relied upon assumptions about the error distributions, but less attention has been accorded to the qualifications Stone made about the knowledge of the error distributions.[25] Consequently, most of the objective functions proposed for use when estimating data matrices have relied upon strong assumptions about the error distributions (typically that they are normally distributed) that are not based on evidence.

Instead of making very strong assumptions about error distributions, Shannon information theory assumes that the probability distribution is to be estimated in a Bayesian framework. Starting with an informative or uninformative prior, information in the form of knowledge about moments of the distribution is used to estimate a posterior distribution. Shannon starts by specifying that the entropy function of a discrete probability distribution is the appropriate measure of information:

$$H = -\sum_k p_k \, log(p_k) \tag{9}$$

Golan (2018, pp 19-22) demonstrates that the entropy function uniquely satisfies various desirable sets of axioms and that, in estimation, maximizing it defines an uninformative prior so that the solution is the least unlikely or least informed outcome. To include prior information about the probabilities, the maximum entropy (ME) formulation can be generalised using the Kullback-Liebler (Kullback and Leibler, 1951; Kullback, 1954) cross entropy (CE) divergence between the estimated and prior probabilities. Formally the CE, or relative entropy, function can be written as

$$CE(p, p^0) = \sum_k p_k log\left(\frac{p_k}{p_k^0}\right) \tag{10}$$

where p_k^0 are the prior probabilities and p_k are the posterior probabilities. The objective is to minimise the CE function. If the prior is the uniform distribution (all p_k^0 are equal), then minimizing the CE divergence is equivalent to maximizing entropy.

3.2. <u>Shannon Cross-Entropy (CE) SAM Estimation</u>

As discussed above, the CE approach to SAM estimation can work directly with SAM coefficients, treating them as analogous to probabilities. The use of the Shannon CE approach for SAM estimation requires moving from estimating probabilities to estimating SAM parameters. The technique can be applied to both values (t_{ij}) and/or coefficients (a_{ij}), or any mix of them. For convenience, the expression below will focus on SAM values. For SAM estimation, consider a general aggregator function that operates on elements of the SAM and assume that the aggregations include an error term reflecting estimation uncertainty,

$$\sum_{i,j} g_{i,j}^m t_{i,j} = \gamma_m + e_m \tag{11}$$

where $g_{i,g}^m$ are a set of m aggregator functions (if $g_{i,j}^m = 1$ then t_{ij} is included in that aggregate while if $g_{i,j}^m = 0$ then t_{ij} is NOT included in that aggregate), and γ_m is the prior estimate of the

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25 "The conclusion …. is that it is feasible to adjust very large Economic Accounting Matrices when one is prepared to forego knowledge of the updated variance matrix" (van der Ploeg, 1982, p). Now it is feasible to estimate very large SAMs using information about error distributions.

value and e_m is the error term for the aggregate m. There are three broad categories of aggregates. First, they might refer to row/column sums, as in the RAS approach. Second, they might refer to various macro aggregates. Third, they might refer to individual elements of the SAM (as many m's as there are elements to be estimated). The estimation of the γ's embodies Stone's labourious work programme, providing a prior for the entire SAM. Note that the prior need not be consistent, e.g., corresponding row/column sums may not equal.

Golan, Judge, and Miller (1996, Chapter 6) describe a generalized cross-entropy (GCE) estimation procedure for the error terms (e's), moving from probabilities to values. Each error term is written as a discrete probability weighted sum with k^melements (where k may differ for different m's) of an error support set $\bar{V}$:

$$e_m = \sum_{k\in k^m} W_{k,m}\bar{V}_{k,m} \tag{12}$$

where $\bar{V}$ are in units of TVs, $W_{k,m}$ are probabilities to be estimated, and $\sum_{k\in k^m} W_{k,m} = 1$ for all m. Estimating the parameters of the SAM is accomplished by estimating all the W probabilities, which is done by minimizing the CE function:

$$CE(W,\bar{W}) = \sum_{k\in k^m,m} W_{k,m} log\left(\frac{W_{k,m}}{\bar{W}_{k,m}}\right) \tag{13}$$

where the $\bar{W}_{k,m}$ are priors on the probability weights, and constraint equations:

the adding up constraints on the probabilities, $\sum_{k\in k^m} W_{k,m} = 1$, consistency constraints on the SAM, $\sum_i T_{i,j} = \sum_i T_{j,i}\ for\ all\ j$, and the aggregator functions.

The number of elements, k, in the error support sets determine the extent to which the properties of the estimated (posterior) error distribution can be recovered from data. For example, for k=3, the prior on the error distribution ($\bar{W}$) may incorporate information about the mean and variance of the distribution, and the posterior will have two degrees of freedom in estimation. If k=5, the prior may incorporate information on additional moments, e.g., skewness and kurtosis, and the posterior estimate will have two additional degrees of freedom.

In practice, the support set values ($\bar{V}$) are chosen to span the possible domain of the errors. If we assume that we start with a prior on the standard error of measurement (s) expressed as a fraction, e.g., 0.1 or 10%, times the prior value of the parameter γ, then the domain of the error support set (for each e_m) will be specified as spanning the domain -3s to +3s where γ is the prior value. Minus to plus three standard errors is a conservative prior domain. The k values of $\bar{V}$ will be spread evenly over the domain, e.g., for a three-element support set k, the values of $\bar{V}$ will be (-3s, 0, +3s). For a five-element support set, they will be (-3s, -1.5s, 0, +1.5s, +3s).

The priors on the probability weights ($\bar{W}$) determine how informative is the prior error distribution, e.g., for an uninformative prior, we typically specify a seven-element support set (k) with $\bar{V}$ values of (-3s, -2s, -1s, 0, +1s, +2s, +3s) and uniform priors ($\bar{W}$) all equal to 1/7. The errors can be anywhere in the error domain with equal probability. The prior is informative only in specifying the bounds. With a seven-element support set, the estimated posterior distribution is essentially unconstrained. For a continuous uniform distribution between bounds, the variance is given by:

$$\sigma^2 = \frac{\left(3s - (-3s)\right)^2}{12} = 3s^2 \tag{14}$$

For the uniform discrete probability distribution, with a seven-element support set, the prior variance equals $4s^2$, so the discrete distribution provides a conservative uninformative prior.

In SAM estimation, we specify two possible informative priors: (1) a general two-parameter distribution with mean zero and variance $\sigma^2 = s^2$, which requires a three-element support set, and (2) a normal distribution with mean zero, $\sigma^2 = s^2$, skewness = 0, and kurtosis = $4\sigma^4$, which requires a five-parameter support set. For the two-parameter distribution, the weights for the discrete prior can be calculated given the mean of zero, known variance, and symmetry:

$$\bar{W}_1 = \bar{W}_3 = \frac{1}{18} \; and \; \bar{W}_2 = \frac{16}{18} \tag{15}$$

For the prior normal distribution, the prior weights can be calculated for the discrete distribution given known mean, variance, skewness, kurtosis, and symmetry:

$$\bar{W}_1 = \bar{W}_5 = \frac{1}{162} \; ; \; \bar{W}_2 = \bar{W}_4 = \frac{16}{81} \; ; \; \bar{W}_3 = \frac{48}{81} \tag{16}$$

Both informative priors have a significant prior weight at the mean, which is zero error.

The CE estimation function is a compound function that supports simultaneous estimation of transaction values given priors on all SAM elements, row and column control totals, and a general set of sub aggregates, as well as standard errors of estimation. The aggregator function presented above specified SAM values and prior additive errors. The general approach accommodates SAM elements expressed as either values or coefficients, and the error term can be specified as either additive or multiplicative. For multiplicative errors, the prior on the standard error of measurement (s) is unitless, e.g., plus/minus some multiplicative value, and the prior discrete error distributions are logarithmic. Estimates of parameters with multiplicative errors are sign preserving, while estimates with additive errors can change sign.

The estimation method is designed to make efficient use of all available information making as few assumptions as possible about missing information, while accepting that some priors and assumptions must be made for missing information.[26] . It addresses all the standard problems associated with the estimation of economic data matrices: it uses all available information but does not impose uninformed constraints on the system to ensure a solution[27] Hence, the approach is a general stochastic specification, as requested by Thorbecke.

3.3. Shannon Cross-Entropy (CE) SAM Estimation: Practical

The GAMS code (SAMEST) for this implementation of Shannon cross entropy SAM estimation, together with Technical Document and a User Guide (McDonald and Robinson,

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26 If there is no information about specific transactions that are known to have taken place but there is no data, then the transactions are entered as zero. The resultant matrix will necessarily be distorted. Introducing an assumption that those transactions were nonzero, with a placeholder entry, allows specification of a complete and consistent matrix.

2023a and b), are open source (http://cgemod.org.uk/samest.html). A prior implementation is illustrated in Robinson (2013), cesam2.gms, is available as part of the GAMS program. [28]

The Shannon CE approach requires simultaneous estimation of many probabilities (support sets for all SAM elements and all control aggregates). With modern algorithms[29] it is practical to use SAMEST to estimate large SAMs/matrices, e.g., 300 and more accounts. In practical applications it can be helpful to derive initial estimates assuming uninformative priors and then refine the estimates by adding informative priors where appropriate. For large SAMs with an ill conditioned prior SAMs, especially, a process of sequential estimation using "progressively disaggregated priors may be appropriate, since with each solution an examination of the resultant marginals indicates those prior TVs for which better prior estimates would be valuable. For very large multi-region SAMs, e.g., global 3-dimensional SAMs, a process whereby inter-regional matrices, e.g., bilateral trade flows, are estimated and the TVs are fixed prior to estimating the intra-regional TVs is a viable option.

This estimation method encompasses all five of the categories of information that have been recognised in the literature since the 1940s. The estimation procedure uses the prior estimates of transactions and control totals that can be subject to different degrees of uncertainty, from very uncertain to certain, with different prior error distributions, from uniform to normal, based on evidence and/or subjective judgement. The ability to estimate SAM data in a matrix in value or coefficient form with additive or multiplicative errors not only provides flexibility it also addresses the potential problems arising with negative TVs without a need to reorganise the accounts in the matrix. Similarly, the flexibility with which aggregate control totals can be incorporated avoids any need to reorganise the accounts in the matrix and simplifies the use of control totals. The option to impose an uninformative prior, with potentially wide error bounds, on any transaction provides a method for addressing the problems of missing information with unrestrictive assumptions (e.g., estimating an inventory investment account).

4. Concluding Comments

All economywide models are based on national accounts data, which are in turn based on receipt/expenditure data for all economic actors. A Social Accounting Matrix (SAM) displays the accounts in a table format, with column expenditures and row receipts. A SAM is a convenient way to organize the accounts for estimation. SAM estimation draws on information from a variety of sources at various levels of aggregation and is bound to involve problems of empirical inconsistency with accounting principles that underlie any economic accounting system (e.g., receipt/expenditure balance for all agents). All national accounts and SAM data, in whatever framework, are subject to sampling and measurement errors because, in large part, they are the products of underdetermined estimation problems. In this environment, standard statistical estimation methods that assume lots more data than parameters to be estimated are of limited use.

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28 The general aggregation function used above is a compact algebraic representation of the method. In the GAMS code, the objective function includes various components because the control totals are subdivided into groups: SAM elements, row/column controls, macro-SAM controls, and miscellaneous other controls.

29 Examples for GAMS include CONOPT 4, KNITRO and PathNLP.

Information theory, which defines uniquely appropriate entropy and cross-entropy measures of information, provides an excellent framework for SAM estimation. There are two approaches used in estimation. In the first, the entropy metrics are assumed to reflect a broad descriptive concept of information and are applied to positive variables such as SAM elements measured in various units such as values, coefficients, or shares. In the second approach, closely following work by Shannon (1948), the entropy metrics are used in the estimation of probability distributions. This approach is Bayesian in that it relies on an efficient information processing rule that uses all the information available in a Bayesian framework (moving from prior to posterior estimates of probabilities), and does not use, or make assumptions about, information that is not available.

This estimation method encompasses all five of the categories of information that have been recognised in the literature since the 1940s. The estimation procedure uses the prior estimates of transactions and control totals that can be subject to different degrees of uncertainty, from very uncertain to certain, with different prior error distributions, from uniform to normal, based on evidence and/or subjective judgement. The ability to estimate SAM data in a matrix in value or coefficient form with additive or multiplicative errors not only provides flexibility it also addresses the potential problems arising with negative TVs without a need to reorganise the accounts in the matrix. Similarly, the flexibility with which aggregate control totals can be incorporated avoids any need to reorganise the accounts in the matrix and simplifies the use of control totals. The option to impose an uninformative prior, with potentially wide error bounds, on any transaction provides a method for addressing the problems of missing information with unrestrictive assumptions, e.g., estimating an inventory investment account.

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