# Estimation of a transportation model using a mathematical program with equilibrium constraints

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In this paper, it is shown how a mathematical program with equilibrium constraints (MPEC) can be used to estimate the parameters of a transportation model using an inconsistent set of observed prices and transportation costs. Supply and demand quantities in the markets are known but transport flows are unknown. The suggested methodology improves upon previous approaches as it avoids discarding valuable information in the process. Significant numerical problems with gradient solvers lead to the development of an algorithm that handles the specific structure of the employed optimization model. The stability and computational speed of this algorithm is tested and compared to alternative approaches using simulation techniques. The method is then successfully applied to a transportation model for agricultural crops for the country of Benin.

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## 1. Introduction

The estimation problem considered in this paper arises in the context of a multiregional (83 regions) agricultural sector model for the West African country Benin, called BenImpact (see IMPETUS 2003 or Britz and M'Barek 2003). This model is developed to allow projections for agricultural production, water use, and trade flows in the context of a larger multidisciplinary project. Currently (spring 2004) BenImpact features regional supply and demand specifications and spatial trade between regions for 8 primary crop products. Trade of the homogeneous products is assumed to be cost minimising. The problem considered here is how to specify the parameters of this cost minimisation problem based on the observations of transportation costs and regional prices.

The observations of prices and costs are mutually inconsistent in the sense that observed transportation costs do not reflect observed regional price differences assuming homogeneous goods and cost minimising traders. Among several reasons for inconsistency are the existence of transaction costs, aggregation biases for prices, measurement errors, and misspecification of the transportation cost function assumed. The objective of this paper is to balance the dataset for the baseline demand and supply quantities, i.e. to estimate a consistent set of prices and transportation costs which calibrates the base year observations on regional supply and demand.

A rich strand of literature from the early publications of Judge and Wallace (1958) and Takayama and Judge (1964) up to more recent contributions of Litzenberg, McCarl and Polito (1982), Peeters (1990), Kawaguchi, Suzuki and Kaiser (1997) and Guajardo and Elizondo (2003) – to name just a few – deal with similar models. However, those contributions perform no balancing of the baseline dataset at all, i.e. start at a disequilibrium situation, or compute regional baseline prices using the first order condition for the baseline being an optimum, i.e. the dual values of market balance constraints. The latter variant was previously used in BenImpact. The advantage is that the process is fast and numerically stable, and the solution obtained is an equilibrium solution to the transportation cost minimisation problem. The disadvantage is that all available price information except one enumerator price (e.g. the price in some important region or a price average) is discarded and replaced by the dual values. This is unsatisfactory as long as we have reason to believe that the price observations contain valuable information.

We propose a method that allows finding mutually consistent regional prices, transport costs, and trade flows using all available information. Mutual consistency is defined as jointly satisfying the conditions of a transport cost minimum including market clearing conditions. One could also interpret the method as one estimating the parameters of a cost minimisation model, where the estimating equations represent first order conditions of the cost minimisation problem.

This creates a bi-level optimization problem which in turn can be expressed as a mathematical program with equilibrium constraints (MPEC), the solution of which is generally difficult. Therefore, a considerable part of the paper deals with the development and validation of a solution algorithm in the specific context.

The paper is organized as follows. In section 2 we quickly review the standard transportation cost minimisation problem employed in BenImpact and derive the estimation problem and section 3 contains some notes on data sources. Section 4 reviews the literature on bi-level programming and MPEC's, and section 5 some approaches to their solution. Section 6 introduces and validates the solution algorithm. Subsequently, results for the BenImpact model are presented and discussed. After showing that a global optimum exists for the incumbent problem we finally conclude and identify directions of further research.

## 2. Estimation Problem

The simple primal transport cost minimisation model for one product employed in BenImpact assumes a fixed cost rate per quantity unit and can be defined as

$$\min_{\mathbf{x}|\mathbf{c}} \sum_i \sum_{j \in T_i} x_{ij} c_{ij} \tag{1}$$

s.t.

$$s_i - d_i + \sum_{j \in T_i} \left( x_{ji} - x_{ij} \right) = 0 \tag{2}$$

where

| | | |
|---|---|---|
| $T_i$ | = | the set of admissible transport flows from $i$ |
| $x_{ij} \geq 0$ | = | trade flow from region $i$ to region $j$ for $c$, $i,j \in \{1,2,\ldots,n\}$ |
| $s_i$ | = | supply in region $i$ |
| $d_i$ | = | demand in region $i$ |

The first order conditions of this problem are then given by

$$s_i - d_i + \sum_{j \in T_i} \left( x_{ji} - x_{ij} \right) = 0 \tag{3}$$

$$c_{ij} - \left( p_j - p_i \right) \geq 0 \quad \perp \quad x_{ij} \geq 0 \qquad \forall i, j \mid j \in T_i \tag{4}$$

with

| | | |
|---|---|---|
| $p_i$ | = | dual value of market balance in region $i$ |

The way the market balance is written allows positive as well as negative dual values. This is OK since each equation 4 contains the difference between two prices. This means that there are enough degrees of freedom to add a term, uniform across regions, to all prices. This term is here referred to as the *enumerator price* and is denoted by $\delta$. In the estimation below, $\delta$ is treated as a variable. Introducing deviations of fitted from observed values (error terms), the formulation of the estimation problem is

$$\min_{\hat{c}_{ij},\hat{p}_i,x_{ij}} F\left(\left(p_i - \hat{p}_i - \delta\right),\left(c_{ij} - \hat{c}_{ij}\right)\right) \tag{5}$$

s.t.

$$s_i - d_i + \sum_{j \in T_i}\left(x_{ji} - x_{ij}\right) = 0 \tag{6}$$

$$\hat{c}_{ij} - \left(\hat{p}_j - \hat{p}_i\right) \geq 0 \quad \perp \quad x_{ij} \geq 0 \qquad \forall i, j \mid j \in T_i \tag{7}$$

$$\hat{c}_{ij} = \hat{c}_{ji} \quad \forall i, j \tag{8}$$

with

*F(.)* = some criterion weighting and aggregating deviations from observed values, for example weighted least squares

$\hat{p}_i$ = estimated price in region *i*

$\hat{c}_{ij}$ = estimated transport cost per quantity unit from region *i* to *j*

Expressed in words, this optimization model searches for the set of transport costs, transport quantities, and regional prices satisfying i) that transport costs and regional prices are as close as possible (measured by some criterion function) to the observed values, while ii) prices and transport flows constitute an optimal solution to the transport cost minimisation problem at the estimated transport costs. In the current application, a weighted least squares criterion was used, weighing price deviations with the inverse of the number of prices and transportation costs with the inverse of the number of transportation costs to estimate.

The optimization problem defined by equations 5 to 8 falls into the class of problems called *mathematical programs with equilibrium constraints* (MPEC). In this case, where equations 6 and 7 represent first order conditions of another problem, the program can also be called a *bi-level program*.

## 3. Data

Only noisy information on transportation costs between the regions and market prices is available: The transportation costs $c_{ijo}$ from region *i* to region *j* of product *o* were computed using the function $c_{ijo} = LRT + FRT_o DIST_{ij} DISC^{DIST_{ij}}$ where *LRT* is a fix cost component, $FRT_o$ is the product specific freight rate, $DIST_{ij}$ is a table of distances and *DISC* is a discount factor decreasing the transportation cost per kilometre for longer distances. In the current paper, no attempt is made to estimate the parameters of this function, but the values of $c_{ij}$ resulting from these calculations are taken as observations (and are in the following referred to as such). Price observations were available monthly but unfortunately only for a different regional aggregation. In the absence of monthly quantity data, the annual price observations used are computed as arithmetic means, and mapped onto the regions of the model from the (mostly larger) regions in the price statistics. No (or little) reliable information on transport flows within Benin is available from official statistics. In the discussion at the end of this paper we discuss an alternative model that makes better use of data.

## 4. About bi-level programs

A bi-level program is an optimisation problem, called the *outer problem*, which uses the solution of another optimisation problem, called the *inner problem*, as its domain. The problem formulated in equations 5 to 8 of the previous section can be formulated as a bi-level program, with the outer problem minimising the weighted squared deviations from observed values and the inner problem minimising transportation costs. Early applications of bi-level programs are reported in the literature on game and principal-agent theory. An example is the Stackelberg game, where a leader (principal) selects an optimal strategy maximising profit or utility subject to technology, but also subject to the optimal choice of the followers who take the leader's actions as given (Luo et al. 1996).

There are very few applications of MPEC in the area of agricultural economics, perhaps because such problems either are difficult to solve or can be reduced to ordinary single level programs. Hazell and Norton (1986) formulate, but do not solve, a bi-level problem for agricultural policy makers. The inner problem describes the behaviour (profit maximisation) of the agricultural sector under the given policy. In the outer problem, the policy maker optimises a "policy objective function" that depends on the policy selected and the optimal behaviour of farmers, the latter of which is not directly controlled by the policy maker.

Bard et al. (2000) formulate a bi-level program to help policy makers arrive at rational policies for encouraging biofuel production. The outer problem is to minimise expenditures for the policies, whereas the inner problem is a linear programming problem describing the profit maximising behaviour of 393 farms in France. Candler et al. (1981) provide a review of the use of multi-level programming in agricultural economics up to 1980. Although somewhat old, their paper still covers many important contributions. However, no publication until now uses a bi-level program to estimate the parameters of a transportation model.

Heckelei and Wolff (2003) use bi-level programs in a way that comes closest to the application considered in this paper. Using a generalized maximum Entropy criterion (outer problem), they estimate the parameters of various versions of agricultural supply models where the estimating equations represent first order conditions of the assumed optimizing behaviour of farmers (inner problem). For one example they also include a complementarity constraint. They do not report any numerical solution problems despite using gradient based solvers which might be due to the fact that their illustrative simulations are based on small models with generated data. This is not true for the transportation cost problem at hand and consequently, the following section suggests the use of a special algorithm after giving a review of previous approaches used.

## 5. Solving the bi-level program

There are different approaches available for solving a bi-level program, all of which have in common that they do not work equally well for all types of problems in this class. It lies neither within the scope of this paper nor within the field of competence of the authors to provide a thorough review of the solution methods reported in literature. Nevertheless, a brief overview is necessary.

Using standard optimisation packages, one general option is to use results of explicit optimisations of the inner problem in the solution of the outer problem. Hazell and Norton (1986 pp 321) recommend a procedure that involves first solving the inner problem for different values of the outer problem variables (e.g. solving a sector model under different policy assumptions), and then using this information as restrictions to the outer problem. Bard et al. (2000) use two approaches that are related to the approach of Hazell and Norton: (i) a grid search over the available design space that stepwise refines the solution, and (ii) an approximation that reduces the problem to an NLP one-level problem, using information from repeated solutions of the inner problem to define behavioural functions. Neither of those approaches seems applicable for the incumbent problem, because the space of the design variables (transportation costs) has too many dimensions (6036). Stepping through this space in a systematic way would require an exorbitant number of iterations.

Another approach is to convert the bi-level problem into a single level non-linear program, where the inner problem is defined by its first order conditions, i.e. equations 5-8. This places the problem into the class of mathematical programs with equilibrium constraints (MPEC) which is perhaps better understood as a mathematical programming problem constrained by at least one complementarity constraint or variational inequality (Harker and Pang 1988, Luo et al.1996). In our case, this reformulation can be done by formulating the complementary slackness as a multiplication equalling zero. Our problem to estimate is then:

$$\min_{\hat{c}_{ij},\hat{p}_i,x_{ij}} F\left(\left(p_i - \hat{p}_i - \delta\right),\left(c_{ij} - \hat{c}_{ij}\right)\right) \tag{9}$$

s.t.

$$s_i - d_i + \sum_{j \in T_i}\left(x_{ji} - x_{ij}\right) = 0 \tag{10}$$

$$\hat{c}_{ij} - \left(\hat{p}_j - \hat{p}_i\right) = \pi_{ij} \quad \forall i,j \mid j \in T_i \tag{11}$$

$$\pi_{ij} x_{ij} = 0 \qquad \forall i,j \mid j \in T_i \tag{12}$$

$$x_{ij} \geq 0 \tag{13}$$

$$\pi_{ij} \geq 0 \tag{14}$$

$$\hat{c}_{ij} = \hat{c}_{ji} \quad \forall i, j \tag{15}$$

Nevertheless, a general MPEC is an "extremely difficult optimization problem" (Luo et al.1996). Complexity can arise from several sources, where the two most important ones in the current study turn out to be that (i) the complementarity conditions cause combinatorial problems when determining which constraints are to be satisfied as equality and which as inequalities, and (ii) the solution space defined by the first order conditions of the inner problem may be non-convex.

The incumbent problem has a non-convex feasible space, as is easily seen in equation 12: If two feasible points exist where one is characterized by $x_{ij}>0$ and the other by the corresponding $\pi_{ij}>0$, there are no feasible points on a straight line between those points. If for example $(x,\pi) = (1,0)$ and $(x^*,\pi^*) = (0,1)$ are two feasible points, the point $(tx+(1-t)x^*,\ t\pi+(1-t)\pi^*)$ cannot be feasible for any $0<t<1$. Therefore, a direct solution of the model with a gradient based solver is bound to find one of several local optima, but has little chance of finding the global optimum. The solution found depends upon the starting values.

The combinatorial nature of MPEC has led some authors (e.g. Gümüs and Floudas 2001) to propose mixed integer programming methods, where binary variables determine which restrictions are to be binding and which not. Tests have indicated that this approach is unsuitable for the problem at hand. Yet another method (e.g. Scholtes and Stöhr 1999) is to remove the complementarity restriction and instead represent it by a penalty function as part of the objective function that – roughly speaking – makes any solution violating the (now removed) complementarity constraint suboptimal.

Ferris et al. (2002) provide a convenient GAMS tool for solving MPEC by smooth NLP reformulations that stepwise approximate the real problem.[2] The algorithm suggested in this paper can be viewed as an extension of one of their approximations. They propose introducing the auxiliary parameter $\mu > 0$ which is used to define a relaxed complementarity slackness condition. For the problem at hand, one of these reformulations (the one that worked best in terms of computational speed and reliability of results) is the following expression substituting for equation 12

$$\pi_{ij} x_{ij} \leq \mu \qquad \forall i, j \mid j \in T_i \tag{16}$$

They then propose that $\mu$ is stepwise decreased, in each step keeping the solution from the previous step as starting value for the next step, and finally set to zero, recovering the original MPEC. This method, though feasible, requires a considerable

[2] The NLPEC (beta release) software shipped with GAMS reads models in MPEC format and reformulates them as ordinary non-linear models that can be solved with conventional NLP solvers. The software supports several different reformulations and a simple algorithm for stepwise reducing the approximation error. The above reformulation was inspired by one of the reformulations available in NLPEC

amount of computation time for the problem at hand, especially when $\mu$ is set to a small value.

We observe when using proposed reformulation that, (i) the program solves faster when $\mu$ is large, (ii) there is little loss in objective value as $\mu$ approaches zero, and (iii) considerable computation time is spent searching for a feasible point. This led us to generate instead the first feasible point using the dual values of an explicit solution of the inner problem based on observed transportation cost and only use two sequential runs of the calibration problem skipping the complementarity restrictions altogether (i.e. implying $\mu = \infty$) in the first and introducing them strictly (i.e. implying $\mu = 0$) in the second. The complete algorithm can be summarized by the following four steps:

Step 1) Solve the inner problem (transport cost minimisation) based on the observed transportation costs to obtain consistent prices (for smaller versions of this problem, this step can be omitted)

Step 2) Solve a "relaxed" version of the MPEC using the feasible starting values generated in step 1 (i.e. initialise prices to be estimated with shadow values of market balances). The relaxation is obtained by simply omitting the complementarity constraint, i.e. solving the problem consisting of equations 9-11, 13-15.This renders a problem that is smooth and convex.

Step 3) Solve the inner problem again, using the transport costs from the solution of step 2.

Step 4) Initialise prices with shadow values of the market balances from step 3. Solve the system of equations 9-15 with an NLP solver to obtain the local optimum. We claim that the solution found is close to the global optimum.

The proposed algorithm solves faster than the algorithm provided with NLPEC even if the latter is applied such that only two steps (with $\mu >> 0$ and then $\mu = 0$) are used, because

i) solving the inner problem explicitly before attempting to solve the MPEC gives a feasible starting point. Since the inner problem is an LP, solving it is much faster than finding a feasible point for the MPEC based on the gradient search methods typically implemented in non-linear solvers.

ii) In our step 2, the "difficult" set of complementarity constraints is left out, making the problem solve faster compared to a solution with only a partially relaxed version of the constraint (with $\mu >> 0$).

It should be noted here that for a more complex or even larger transportation problem the feasible starting point generated by step 3 might not be close enough to the global optimum for the algorithm to perform well with gradient solvers. However, the results below show that this is not the case in out model context.

## 6. Performance of Algorithm and Analysis of Results

To verify that the algorithm proposed avoids starting value sensitivity and actually finds the global optimum for the type of problem at hand, a smaller version of the problem was constructed where the global optimum is known. The illustrative program had ten regions, the true (symmetric) transport costs were drawn from a uniform distribution in the interval [1,10]. The transport minimisation problem was solved, and the true prices computed as the shadow values of the market balance plus the $\delta$ introduced in the previous section (to avoid negative true prices). Supply and demand were drawn from uniform distributions as well, and demand was scaled with a factor uniform across regions so that total demand matches total supply (the market balance is feasible).

The didactic program was solved from 1000 randomly picked starting points using first a gradient based NLP solver directly applied to the original MPEC and then by applying the algorithm introduced above. The procedure was as follows:

- Loop 1000 times

    a) Reset all variables and dual values.

    b) Generate starting point by drawing transportation costs
    from uniform distribution.

    c) Solve transportation cost minimisation problem with the random
    transportation costs to get a feasible starting values for transport flows,
    and use dual values of market balance to initialise prices.
    $\delta$= average observed price.

    d) Solve the MPEC defined by equations 9-15.

    e) Repeat steps a) to c)

    f) Solve the problem using the algorithm, leaving step 1 of the algorithm out.

    g) Compare the solutions from steps d) and f) with all previous solutions to
    determine if these solutions have already been found.

- Next iteration.

As objective function, weighted least squares was used with the weight $n$ for price deviations and $n(n-1)/2$ for transportation costs, reflecting the number of possible transport flows connected to each region. A solution was deemed unique if the sum of squared differences between current estimated prices or transportation costs and any of the previous solutions was smaller than 0.001.

The test revealed that the algorithm found the global optimum in each of the 1000 draws, whereas the direct approach found 1000 different solutions and never the optimal one (with objective value of zero = no deviations). The distribution of the objective value in the direct approach is shown in diagram 1. This simulation result is promising but obviously does not prove that the suggested algorithm would always succeed in finding the global optimum. The sample program was purely synthetic and easier to solve than the real problem, because it featured far less regions and consequently variables (only 10 regions compared to 83 in the full scale problem, and 282 variables compared to 15175) in order to be able to perform a lot of draws within reasonable computation time. The sample problem was also comparably well behaved in the sense that the observation from which deviation was minimised was a feasible point.

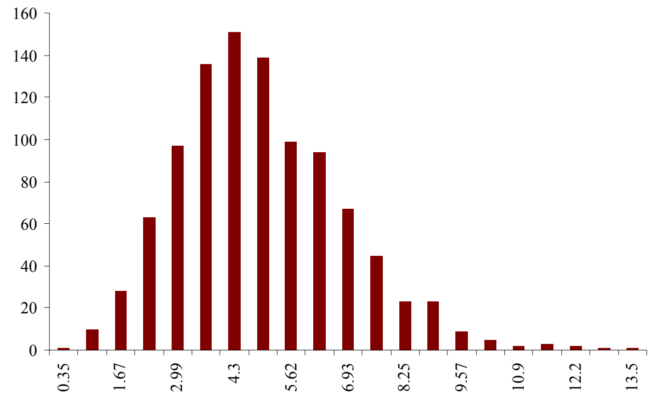


**Diagram 1: Distribution of objective values found with direct NLP approach. Number of solutions per size class.**

We now turn to the real world, full scale problem and test sensitivity to starting values: The problem was solved 100 times, each time with new starting values drawn from a uniform distribution and used to initialise the estimation problem as in the didactic size test. Afterwards, the distribution of the estimated parameters and objective values were analysed. Again, the first step in the algorithm was omitted. The full scale test suggests that the eight products studied can be split into two groups: One group of crops behaves nicely, with stable estimations, whereas the other group behaves in a more problematic fashion.

Table 1 shows the results of the sensitivity analysis for niebe (a bean), which is one of the more problematic products (in terms of the estimation). The first row shows the number of trade flows that is qualitatively different in any of the estimations, i.e. the number of region pairs (*i,j*) for which the 100 simulations did not produce either of the two monotonous series with only $x_{ij} > 0$ or only $x_{ij} = 0$. It is to be expected that this number is rather high, as there are more than 6000 possible trade streams of which only 83 will be realised in a transport cost minimising solution. Several possible transport flows between two regions (possibly over intermediary regions) can have the same or very similar transportation cost, showing that the solution is possibly degenerate. This is disturbing but not a serious problem since all such solutions are, by the chosen criterion function, equally good. More important is to look at the next three rows showing the mean and stability of the objective value. This is important, because the weighted sum of squared deviations is by definition the criterion by which we determine the quality of the solution. The lower the value, the closer the fit of the estimates to observed values. The algorithm performs reasonably well in obtaining a rather stable objective value, even though there is some dependency on starting values also in this regard.

The last four rows show means and coefficients of variation for transport costs and prices. Due to the degenerate character of the transport sub problem, the coefficients of variation for transport costs are higher than those for prices. Prices seem to be stable, with very low coefficients of variation.

**Table 1: Sensitivity analysis for niebe. 100 draws of starting values were performed with the direct solution and with the algorithm.**

| | | |
|---|---|---|
| Trade streams qualitatively changed in any iteration (i.e. x=0 → x>0 or reverse) | | 354 |
| Objective: | mean | 42.99 |
| | sample std. Dev. | 0.295 |
| | minimum value | 42.409 |
| | maximum value | 43.587 |
| Transport cost: | mean** | 4.765 |
| | coeff. of variation* | 0.271 |
| Prices: | mean | 120.602 |
| | coeff. of variation | 0.004 |

** The coefficient of variation was computed based on the relative deviation of each item from its series mean.*
*** No weighting with transport flows. Transport costs for flows that are zero are also counted.*

The estimation of cassava behaves clearly better. The results of the sensitivity analysis, shown in table 2, indicate that there is no starting value dependency regarding the estimation of prices, but some for the transportation costs. The objective value reached was identical in all iterations as far as the machine precision allows to determine this. The number of trade streams that change qualitatively is higher than for niebe.

**Table 2: Sensitivity analysis for cassava. 100 draws were performed with the direct solution and with the algorithm.**

| | | |
|---|---|---|
| Trade streams qualitatively changed in any iteration (i.e. x=0 → x>0 or reverse) | | 1239 |
| Objective: | mean | 124.552 |
| | sample std. Dev. | 0 |
| Transport cost: | mean** | 7.137 |
| | coeff. of variation* | 0.034 |
| Prices: | mean | 97.831 |
| | coeff. of variation | 7.31E-6 |

** The coefficient of variation was computed based on the relative deviation of each item from its series mean.*
*** No weighting with transport flows. Transport costs for flows that are zero are also counted.*

There appears to be a qualitative difference between the behaviour of niebe and cassava: When using the algorithm, step 2, the less constrained model, delivers almost precisely the same objective as step 4, the full model, for cassava, whereas for niebe the objective in step 2 is significantly lower than that in step 4. This seems to depend on whether the price data show higher prices in deficit regions (consistent with transport cost minimisation under free trade) or not. If the observed prices are higher in deficit regions than in excess supply regions, and also higher than transportation costs between the respective regions, the complementarity restrictions will be close to feasible without being imposed. If observed prices do not show a tendency towards these properties, the objective function values gains from letting $\pi$ be big also between regions where there is a transport flow. Of the eight primary products currently featured in BenImpact, three behave more like niebe and the other five like cassava. This indicates that better data would (as usual) improve the estimation. Another explanation of this phenomenon would be that for some products like cassava, the number of degenerate solutions is larger than for others. This would be supported by the higher number of qualitatively different transport flows between different estimates as indicated in Table 2. A higher number of degenerate solutions makes is more likely to find a solution close to the global optimum for the different starting values already in the "relaxed" estimation step 2.

It is not obvious that a global optimum exists for the estimation problem presented in this paper. If that is not the case, it would always exist a better solution than any one already found (because we can find at least one candidate for a global optimum). Indeed, Lou et al. (1996) demonstrate that an MPEC may fail to comply with the sufficient conditions for a global optimum to exist. In the following section we show that the incumbent problem has a global optimum.

## 7. Existence of a global optimum

The Weierstrass theorem (Intriligator 1971) says that a continuous function $F(s)$ has a maximum and a minimum for some $s \in S$ if $S$ is closed and bounded. Applied to the problem at hand, this requires that (i) the objective function (squared deviations) be continuous and (ii) the feasible set $S = \{(p,c) | c \in C, p \in \Gamma c\}$ be closed and bounded.

Harker and Pang (1988) use this to show that for an optimum to exist, it is sufficient that

(A) The set $S$ is nonempty and closed

(B) There exist a scalar $\alpha > 0$ and a feasible vector $(u,v) \in S$ with $\|(u,v)\| \leq \alpha$ such that $F(c,p) \geq F(u,v)$ for all $(c,p) \in X$ with $\|(c,p)\| \geq \alpha$.

B means that the search is limited to a closed ball around origo in $S$, and it certainly holds if we can find any feasible point, because the objective function F is *strictly convex*. Since F is strictly convex, we can, if given any feasible point $(u',v')$ in $S$, limit our search to the closed convex subspace $S' \subseteq S$ satisfying $F(c,p) \leq F(u',v')$, i.e. points that are at least as good as $(u',v')$. The closed ball required in B is then found by taking the smallest ball that contains $S'$. Finding the feasible point $(u',v')$ is easily done since $C \neq \emptyset$ and $\Gamma c \neq \emptyset$ for all $c \in C$. Because of the latter conjecture (which we leave to be believed without proof), we can pick for example the observed $c$. Note that it is not necessary to show that the space is closed also in the dimensions of the transport flows, since they are not part of the objective function.

This leaves us to show that *S* is closed, which can be shown as follows:

According to Berge (1997p. 111), *S* is a closed set if and only if Γ is a closed mapping. A mapping from *C* to *P* is closed if whenever $c_0 \in C$, $p_0 \in P$, $p_0 \notin \Gamma c_0$ there exists two neighbourhoods $U(c_0)$ and $V(p_0)$ such that $c \in U(c_0) \Rightarrow \Gamma c \cap V(p_0) = \emptyset$. In words this means that if a price $p_0$ lies outside the solution set for some transportation cost $c_0$, then $p_0$ can not be a *limit* of the solution set, i.e. it is possible to find more prices close to $p_0$ in any direction that are also not part of the solution; the solution's complement is open. The first order and complementary slackness conditions for a solution to the transportation cost minimisation problem are the mapping from costs to prices. That mapping is closed, because the inequalities involved always take the form of ≤ or ≥, meaning that the limiting points are feasible. This can also be seen by a didactic size example where the possible solutions are stated explicitly:

There are three regions, A, B and C as shown in diagram **1**. A has a surplus 1 and C has a deficit 1, whereas B has neither surplus nor deficit. The costs of transportation are called $c_{AB}$, $c_{BC}$ and $c_{AC}$ (and the same costs for the reverse streams). Flows are denoted $x_{AB}$, $x_{BC}$ and so on. There are now three possible ways of satisfying the market balances:

(1) $c_{AC}<c_{AB}+c_{BC}$. Then $x_{AC}=1$ and all other flows =0, $p_C=p_C-c_{AC}$ and $p_A+c_{AB}\geq p_B\geq p_C-c_{BC}$.

(2) $c_{AC}>c_{AB}+c_{BC}$. Then $x_{AC}=0$, $x_{AB}=x_{BC}=1$, $p_B=p_A+c_{AB}$, $p_C=p_A+c_{AB}+c_{BC}$.

(3) $c_{AC}=c_{AB}+c_{BC}$. Then (1) and (2) are identical solutions and trade can flow directly to C from A or over B or both.

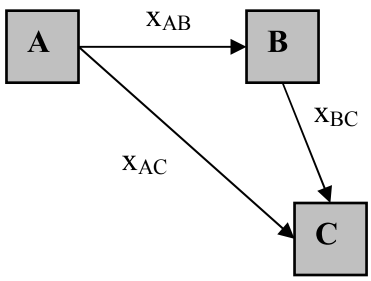


**Diagram 1: Regions and trade flows.**

In all three cases, the set of possible prices is closed, and hence Γ is a closed mapping and the set (image) *S* is also closed. The reader may see that there is one degree of freedom for prices in the conditions (1) and (2), meaning that if *p* is a price vector that solves the problem, then so does $p'=p+\delta$ where $\delta>0$. Note however, that by condition (B), there is no chance of finding better solutions by letting $\delta\rightarrow\infty$.

To conclude up: conditions (A) and (B) can both be shown to hold, and hence a global optimum exists.

## 8. Conclusions

The paper suggested an approach to estimate the parameters of a transportation model using inconsistent observations on prices and transportation cost which avoids information loss inherent in previous approaches. A mathematical program with equilibrium constraints has been employed which minimises weighted squared deviation of estimated from observed values subject to complementarity restrictions that guarantee a solution representing a transport cost minimum. Using a four-step algorithm, the estimation problem could be solved with reasonable computation time. The solution depends slightly upon starting values, i.e. the global optimum is not necessarily found with all data constellations, but sensitivity analyses show that the solution is most likely to be close to the global optimum, and for simple cases where the global optimum is known, that optimum is also found exactly..

The estimation performed in this paper leaves room for improvements and further research:

- The parameters of the function used to generate transportation costs are treated as exogenous. This equation could be included and estimated simultaneously with prices and transportation cost .
- Price observations may have an aggregation bias, because monthly prices are just averaged up to annual prices.. In an upcoming research project we will attempt to consider supply fluctuations by making the model a multi-period model.
- In this paper, no attention is paid to trade flows between regions, simply because we have no information that can be used to rank different flow patterns. However, there is some limited information on trade flows on a regionally more aggregated level. If the model is reformulated so that this level is included, the observations on trade flows may be included in the criterion function as well.

Taking a more general perspective, the results of this research show that by formulating estimations of optimisation models as MPEC, the parameters of that model can be estimated in a way fully consistent with the model for which they are going to be used. However, it also confirms that MPEC's are difficult to solve and that each problem may need a tailor made solution method. An important question for future research seems to be how to measure the quality of such estimations, as standard test statistics are not directly applicable. Another issue is the development of better solution algorithms that are easy to use for modellers. The NLPEC project currently shipped with GAMS as a beta version is highly interesting and its development will be closely followed by the authors.

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