

Solving Inequality Constrained Problems using GEMPACK

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Abstract

An alternative method, which is completely different from the existing two pass method, is proposed to solve problems with inequality constraints using GEMPACK. The key innovation of the method is perturbation. The new method is easy to implement in GEMPACK. Numerical tests on a number of examples including an example of perfect substitution with capacity constraints and a hypothetical intertemporal general equilibrium model with inequality constraints are presented to demonstrate effectiveness of the method. The method has also been successfully implemented in the Global Trade and Environment Model (GTEM) and in the Monash Multi-Regional Forecasting (MMRF) model for imposing constrained emission-permit trading as well as in the newly developed Treasury Computable General Equilibrium (TCGE) model for imposing non-negative sector-specific rate of return on capital.

Key words: complementarity conditions, inequality constraints, Kuhn-Tucker conditions, and GEMPACK

1. Introduction

Impositions of inequality constraints are often required in many practical models including applied economic models. Generally, inequality constraints are imposed in problems under an optimisation framework. It is well known in theory that the solution of an inequality constrained optimisation problem can be obtained by solving a set of optimality conditions known as Kuhn-Tucker conditions (Chiang, 1984), which consist of equality conditions, inequality conditions and their associated complementarity slackness conditions.

There are many numerical methods available for solving constrained optimisation problems or a set of optimality conditions (simultaneous equations) (Press *et al.*, 1992), given it is generally impossible to solve them analytically. For solving optimisation problems or simultaneous equations in economics, two software packages are widely used, which are the General Algebra Modelling System (GAMS) (Brooke *et al.*, 1987) and the General Equilibrium Modelling Package (GEMPACK) (Harrison and Pearson, 2002). GAMS can solve optimisation problems with inequality constraints using linear or nonlinear programming (LP or NLP) or MCP for mixed complementarity problems (Dirkse and Ferris, 1993; Rutherford, 1993).

However, solving problems with inequality constraints in GEMPACK is not as straightforward as in GAMS. In GAMS, one can simply write inequality constraints in the same way as the equality constraints into the codes, and then use LP or NLP or MCP to solve the constrained optimisation problems. In GEMPACK, however, all equations have to be written in equality format. Hence, in order to solve inequality constrained problems using GEMPACK, one needs to convert all inequalities to equalities. In addition, GEMPACK requires that all functions used in equations are

¹ This paper is a technical paper only. All errors are solely the author's.

differentiable (so that linearization approach can be used). This leads to some difficulties for handling inequality constraints in GEMPACK.

Harrison *et al.* (2002) proposed a method to implement quota constraints (inequality constraints) in GEMPACK – i.e., the two pass method. The method involves two simulations – approximate simulation and accurate simulation with closure swapping, where a number of iterations, i.e., from approximate simulation to accurate simulation then back again to approximate simulation and repeating the process, may be required to get satisfactory results, depending on particular problems. Therefore, it may need long computational time to get a solution. For a large intertemporal general equilibrium model such as GETEC (Pant *et al.* 2005), the base run alone without any inequality constraint may take more than 48 hours to solve on a Itanium 2 (1.6GHz, Intel Itanium processor) computer – consequently, requirement of long computational time by using the two pass method would make solving this type of model computationally infeasible when necessary inequality constraints are needed to impose in the model. In fact, a recent application of the two pass method to the GTEM (Pant 2007) which is just recursive for simulating constrained emission trading clearly demonstrated that the time required for solving the model is much longer than that for solving the model without the constraint.

Furthermore, it has also been found based on a number of numerical examples tested that the effectiveness of the two pass method is limited for complex inequality constrained problems – for example, the two pass method cannot solve several examples presented in this paper. There are also some issues about decomposition of results with complementarity statements when using the two pass method – the decomposition does not add up at the solution from the accurate simulation (see GEMPACK User Documentation, Release 9.0, GPD-5, Chapter 7).

Given above, an alternative method is proposed in this paper, which is easy to implement in GEMPACK and can be applied to general inequality constrained problems with various types and numbers of inequality constraints. Having inequality constraints in the model using the proposed method does not much affect the computational time at all. GEMPACK's approach on decomposition of the results (i.e., the Subtotal function) also works perfectly fine with inequality constraints using the proposed method.

The core steps of the proposed method involve: (i) perturbing the complementarity slackness conditions, using homotopic variables to compensate the perturbation so that the base or initial solution still satisfies the perturbed slackness conditions; (ii) shocking the homotopic variables to zero to get the true solution of the problem which is needed to solve. The method has been successfully tested on a large number of examples. For all of the examples tested, the method gives the results which are identical to those obtained by GAMS, as well as by Excel for those problems which Microsoft Excel-Solver can handle.

The paper is organised as follows. Section 2 describes the proposed method. Section 3 presents a number of examples, including simple examples and complicated examples for intertemporal optimisation problems with both linear and nonlinear inequality constraints. A brief conclusion is then drawn in section 4.

2. Method

Without loss of generality, consider solving a problem with a set of equality, inequality and complementarity conditions in the following:

$$F_i(X_1, X_2, \mathbf{L}, X_N, Y_1, Y_2, \mathbf{L}, Y_I, l_1, l_2, \mathbf{L}, l_K, h_1, h_2, \mathbf{L}, h_L) = 0, \text{ for } i = 1, 2, \dots, N \quad (1)$$

$$G_k(X_1, \mathbf{L}, X_N, Y_1, \mathbf{L}, Y_I) = 0, \text{ for } k = 1, 2, \dots, K \quad (2)$$

$$h_l H_l(X_1, \mathbf{L}, X_N, Y_1, \mathbf{L}, Y_I) = 0, \text{ for } l = 1, 2, \dots, L \quad (3)$$

$$H_l(X_1, \mathbf{L}, X_N, Y_1, \mathbf{L}, Y_I) \leq 0, \text{ for } l = 1, 2, \dots, L, \text{ and} \quad (4)$$

$$h_l \geq 0, \text{ for } l = 1, 2, \dots, L, \quad (5)$$

where X_i , $i=1, 2, \mathbf{L}, N$ are endogenous variables. Y_i , $i=1, 2, \mathbf{L}, I$ are exogenous variables or parameters. l_k for each k can be regarded as the shadow price or Lagrange multiplier of the equality constraint (2), and h_l for each l as the shadow price of the inequality constraint (4). Equation (1) can be regarded as the first order condition of a constrained optimisation for each endogenous variable X_i . Equation (3) is the complementarity slackness condition for each inequality constraint in (4). Equation (5) represents the requirement on sign of the shadow price for each inequality constraint according to the Kuhn-Tucker theorem (Chiang, 1984). F_i for each i , G_k for each k , and H_l for each l are differentiable.

The conditions (1) to (5) represent a general inequality constrained (optimisation) problem. In fact, these conditions can be derived from the optimality conditions of the following general constrained optimisation:

$$\begin{aligned} & \min_{X_i, i=1, \mathbf{L}, N} f(X_1, X_2, \mathbf{L}, X_N, Y_1, Y_2, \mathbf{L}, Y_I) \\ & \text{subject to:} \\ & \text{(i) } G_k(X_1, \mathbf{L}, X_N, Y_1, Y_2, \mathbf{L}, Y_I) = 0, \text{ } k = 1, 2, \mathbf{L}, K \\ & \text{(ii) } H_l(X_1, \mathbf{L}, X_N, Y_1, Y_2, \mathbf{L}, Y_I) \leq 0, \text{ } l = 1, 2, \mathbf{L}, L. \end{aligned}$$

The question is how the general inequality constrained problem, i.e., conditions (1) to (5), could be solved by GEMPACK.

The inequality conditions in equations (4) and (5) cannot be implemented directly in GEMPACK. They are converted to the following equivalent equality conditions:

$$-H_l(X_1, \mathbf{L}, X_N, Y_1, \mathbf{L}, Y_I) = Z_l^2, \text{ for } l = 1, 2, \dots, L, \text{ and} \quad (6)$$

$$h_l = V_l^2, \text{ for } l = 1, 2, \dots, L \quad (7)$$

where Z_l and V_l for each l are intermediate variables whose values are determined by solving equations (1), (2), (3), (6) and (7) jointly. It should be noted that the signs of Z_l and V_l for each l can not be determined in equations (6)-(7), but this does not affect the solution of the actual problem given by equations (1)-(5).

There are total $3L+K+N$ equations (equations (1)-(3) and (6)-(7)) and $3L+K+N$ variables ($X_i, i=1, \mathbf{L}, N; l_k, k=1, \mathbf{L}, K; h_l, l=1, \mathbf{L}, L; Z_l, l=1, \mathbf{L}, L; V_l, l=1, \mathbf{L}, L$). Mathematically, it is possible to solve the $3L+K+N$ variables from the $3L+K+N$ equations. As all of the equations are expressed in equality form, GEMPACK can be used, in principle, to solve them.

However, there are some practical implementation issues, particularly, issues with solving the complementarity slackness conditions (i.e., equation (3)), because the shadow prices are either zero or positive. The standard GEMPACK linearization approach fails on this equation with the reasons below.

Linearizing equation (3) gives:

$$h_l \cdot \left[\sum_{i=1}^N \frac{\partial H_l}{\partial X_i} dX_i + \sum_{k=1}^I \frac{\partial H_l}{\partial Y_k} dY_k \right] + H_l(X_1, \mathbf{L}, X_N, Y_1, \mathbf{L}, Y_I) \cdot dh_l = 0. \quad (3a)$$

Let all h_l be zero (that is, all inequality constraints are non-binding initially) and $H_l(X_1, \mathbf{L}, X_N, Y_1, \mathbf{L}, Y_I) \neq 0$ initially, then the equation (3a) used by GEMPACK at the first step of solving process is actually:

$$0 \cdot \left[\sum_{i=1}^N \frac{\partial H_l}{\partial X_i} dX_i + \sum_{k=1}^I \frac{\partial H_l}{\partial Y_k} dY_k \right] + H_l(X_1, \mathbf{L}, X_N, Y_1, \mathbf{L}, Y_I) \cdot dh_l = 0, \quad (3b)$$

which gives, as $H_l(X_1, \mathbf{L}, X_N, Y_1, \mathbf{L}, Y_I) \neq 0$,

$$dh_l = 0. \quad (3c)$$

This means h_l has no change at the first step, hence h_l remains at zero. At the next step, (3b) remains. There are two possibilities about $H_l(X_1, \mathbf{L}, X_N, Y_1, \mathbf{L}, Y_I)$ at the next step: either $= 0$ or $\neq 0$, depending on shocks. If $H_l(X_1, \mathbf{L}, X_N, Y_1, \mathbf{L}, Y_I) = 0$, (3b) becomes $0 = 0$ which leads to an GEMPACK error – GEMPACK fails to find a solution; if $H_l(X_1, \mathbf{L}, X_N, Y_1, \mathbf{L}, Y_I) \neq 0$, (3c) then holds – this goes back to the previous step. This process repeats for all solving steps, which will result in either $h_l = 0$ or an error in solving, no matter what shocks have been applied. This means, even if some constraints should be binding with sufficient shocks, GEMPACK just ignores them and still gives $h_l = 0$ or fails to solve.

Because of the above, special treatments need to apply to handle the inequality constraint problems in GEMPACK. The two pass method proposed by Harrison *et al.* (2002) provides a way for solving the inequality constraint problems using

GEMPACK. However, this method has a number of drawbacks as discussed earlier. The purpose of this paper is to introduce a new method which lets GEMPACK get away from the above failure or trap and hence give the correct solution.

A natural idea to avoid GEMPACK falling into the trap of the above incorrect solution is to perturb the initial solution away from the true solution which potentially leads GEMPACK into the trap. So, the new method is, at first, to perturb the complementarity slackness condition (i.e., equation (3)) to the following equation,

$$[h_l + e_l][w_l - H_l(X_1, \mathbf{L}, X_N, Y_1, \mathbf{L}, Y_I)] + \{e_l^0 w_l^0 - e_l^0 H_l(X_1^0, \mathbf{L}, X_N^0, Y_1^0, \mathbf{L}, Y_I^0) + w_l^0 h_l^0\}q = 0, \text{ for } l = 1, 2, \dots, L \quad (8)$$

where e_l and w_l for each l are the “small” positive perturbations introduced, which are exogenous and will be shocked to zero; q is another exogenous variable which takes the value of -1 initially and will also be shocked to zero (this is the usual homotopic approach in GEMPACK to make all equations satisfied with the initial solution); and all symbols with a superscript “0” represent the initial values of the corresponding variables. Perturbations can be added into the equations in many different ways. Equation (8) provides just one way for perturbing the system.

The method is then to perturb equations (6) and (7) accordingly as follows:

$$w_l - H_l(X_1, \mathbf{L}, X_N, Y_1, \mathbf{L}, Y_I) = Z_l^2, \text{ for } l = 1, 2, \dots, L, \text{ and} \quad (9)$$

$$h_l + e_l = V_l^2, \text{ for } l = 1, 2, \dots, L \quad (10)$$

Now, instead of solving equations (1)-(3) and (6)-(7), equations (1)-(2) and (8)-(10) are to be solved, with additional exogenous variables e_l , w_l and q which all are to be shocked to zero, together with intended shocks on any other exogenous variables.

Initial solution for the equations (1)-(2) and (8)-(10) is the same as that for the original equations (i.e., equations (1)-(5), or equations (1)-(3) and (6)-(7)) for all variables appearing in the original equations. The intermediate variables Z_l and V_l can be initialized automatically in the TABLO code using formula:

$$Z_l^0 = \sqrt{w_l^0 - H_l(X_1^0, \mathbf{L}, X_N^0, Y_1^0, \mathbf{L}, Y_I^0)} \text{ and } V_l^0 = \sqrt{h_l^0 + e_l^0} \text{ for } l = 1, 2, \dots, L$$

The additional exogenous variables e_l and w_l can be initialized to a reasonably but arbitrarily small number. To reduce the number of additional exogenous variables, one may set e_l and w_l be equal for all l . In fact, in all examples presented in this paper, only a single additional exogenous variable besides q is introduced, that is, all e_l and w_l are set to be same. Having different e_l and w_l for all l is just to make the perturbations more flexible, as the scales of corresponding level variables may be significantly different – hence different perturbations may be necessary. The exogenous variable q is initialised at -1 (homotopic approach).

Having provided an initial solution as above, equations (1)-(2) and (8)-(10) can now be solved by GEMPACK, following the standard linearization approach on all equations, where all additional exogenous variables introduced for the perturbations are to be shocked to zero, together with intended shocks on any other exogenous variables of the model.

Several examples are given below to explain implementation of the method as well as demonstrate effectiveness of the method.

3. Examples

3.1. Max, Min and Abs functions

The first example is to apply the proposed method to calculate ‘max’, ‘min’ and ‘abs’ in GEMPACK.

Let $M = \max(X, Y)$ where X and Y are given, and M is to be determined. It is not difficult to see that M can be determined equivalently by the following equations:

$$[M - X][M - Y] = 0, \quad (11)$$

$$M \geq X, \text{ and} \quad (12)$$

$$M \geq Y. \quad (13)$$

Applying the proposed method, these equations can be perturbed to:

$$[M - X + e][M - Y + e] + [(M^0 - X^0 + M^0 - Y^0)e^0 + [e^0]^2]q = 0, \quad (14)$$

$$e + M - X = Z^2, \text{ and} \quad (15)$$

$$e + M - Y = V^2, \quad (16)$$

where M^0, X^0, Y^0 and e^0 are the initial values of M, X, Y and e , respectively, with $M^0 = \max(X^0, Y^0)$. As it is mentioned earlier, e^0 can be chosen arbitrarily; so let it be equal to 0.01 here. q is initialised at -1 (i.e., $q^0 = -1$). Z and V are initialised by the following formula:

$$Z^0 = \sqrt{e^0 + M^0 - X^0} \text{ and } V^0 = \sqrt{e^0 + M^0 - Y^0} \quad (17)$$

Instead of solving equations (11)-(13), solving equations (14)-(16) jointly using GEMPACK will give the solution of $M = \max(X, Y)$ by shocking e and q to zero (i.e., shock the change of e by $-e^0$, and shock the change of q by $-q^0$) together with any shocks on X and Y . This has been proved by numerical tests.

Similarly, ‘min’ and ‘abs’ can be solved by GEMPACK. In fact, both ‘min’ and ‘abs’ can be converted to a ‘max’ problem, that is, for any X and Y , $\min(X, Y) = -\max(-X, -Y)$; and for any X , $\text{abs}(X) = \max(X, -X)$.

Apart from calculating the above useful functions in GEMPACK, the main application of the proposed method is to solve inequality constrained problems. Inequality constraints are generally imposed under an optimisation framework, which

then lead to their associated complementarity slackness conditions derived from the optimality conditions. So, several examples provided in the following sections are presented as an optimisation problem with inequality constraints to demonstrate the effectiveness of the proposed method for solving complementarity conditions in GEMPACK.

3.2. A simple starting example

The example presented in this section was the first example tested using the proposed method. It is extremely simple and can be solved analytically. Hence, it provides a good starting example to test the method.

Consider the following problem:

$$\min_{X(t), t=1, \dots, 20} \sum_{t=1}^{20} [X(t) - Y(t)]^2$$

subject to: $X(t) \leq 1$ for $t = 1, 2, \dots, 20$,

where $Y(t)$ for all t are exogenous, and $X(t)$ for all t are endogenous.

This problem does not need to be dynamic, where t can be removed from the problem without affecting anything. The intention of including t in the problem is just for easy demonstration of path of the variable from non-binding to binding by the inequality constraint imposed.

For each t , the Kuhn-Tucker optimality conditions for the problem are:

$$2[X(t) - Y(t)] + h(t) = 0, \quad (18)$$

$$h(t)[X(t) - 1] = 0, \quad (19)$$

$$X(t) \leq 1, \text{ and} \quad (20)$$

$$h(t) \geq 0 \quad (21)$$

where $h(t)$ is the shadow price of the inequality constraint.

An initial solution for the above equations is assumed: for all t , $X^0(t) = Y^0(t) = 0.5$ and $h^0(t) = 0$. Introducing perturbations as well as converting the inequalities to equalities as discussed earlier, the equations (19)-(21) become:

$$[h(t) + e][e + 1 - X(t)] + [1 - X^0(t) + h^0(t)e^0 + [e^0]^2]q = 0, \quad (22)$$

$$e + 1 - X(t) = [Z(t)]^2, \text{ and} \quad (23)$$

$$h(t) + e = [V(t)]^2 \quad (24)$$

Now, for the perturbed system given by equations (18) and (22)-(24), $X(t)$, $Y(t)$ and $h(t)$ are initialised at the same values as those for the unperturbed system, q is initialised at -1 (i.e., $q^0 = -1$), e is initialised arbitrarily at 0.01 (i.e., $e^0 = 0.01$), and $Z(t)$ and $V(t)$ for each t are initialised by the following formula:

$$Z^0(t) = \sqrt{e^0 + 1 - X^0(t)} \text{ and } V^0(t) = \sqrt{h^0(t) + e^0}$$

The solution for the original equations (18)-(21) can be obtained by solving the equations (18) and (22)-(24) jointly by shocking e and q to zero using GEMPACK for a given path of $Y(t)$.

Let $Y(t) = 0.5 + 0.1(t-1)$. The solution of the problem is shown in Figure 1, where it can be seen that X hits the bound of 1 from year 6 when Y becomes 1, and h switches from zero to positive accordingly as expected. The result shows that the proposed method gives the exact solution of the problem. It is also checked that the full original Kuhn-Tucker optimality conditions given by equations (18)-(21) are satisfied with the solution obtained by GEMPACK.

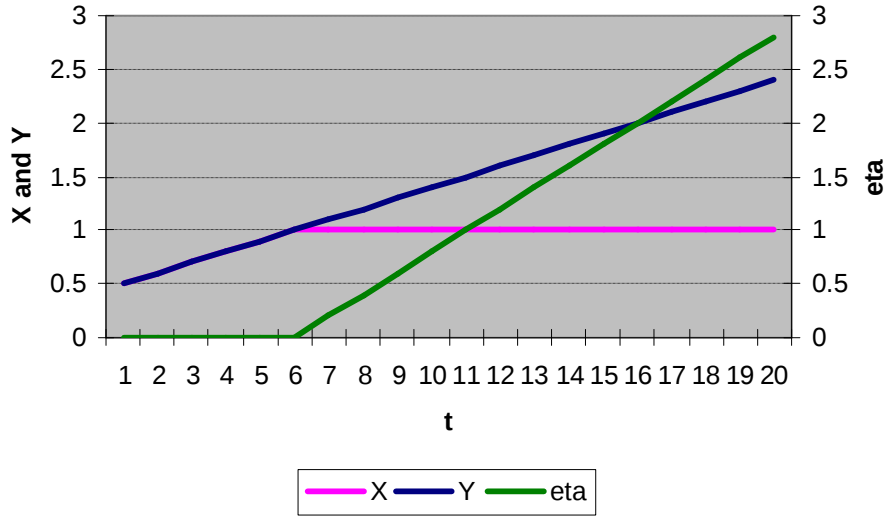


Figure 1. GEMPACK result on the simple example showing the inequality constraint on $X(t)$ binding and its associated shadow price $h(t)$, with the given path of $Y(t)$.

Harrison *et al.* (2002)'s two pass method was also used to solve this example. It gives the same result as above.

3.3. Example of constant elasticity of substitution (CES) function

Consider the following intertemporal optimisation problem:

$$\min_{X(t), t=1, \dots, 20} \sum_{t=1}^{20} \sum_{i=1}^5 P_i(t) X_i(t)$$

subject to:

$$(i) \bar{X}(t) = \left[\sum_{i=1}^5 A_i^{1+r} X_i(t)^{-r} \right]^{-1/r}, \text{ for } t = 1, 2, \dots, 20,$$

$$(ii) X_1(t) + X_1(t+1) \leq 500, \text{ for } t = 1, 2, \dots, 19,$$

$$(iii) X_2(t) \geq 50, \text{ for } t = 1, 2, \dots, 20,$$

$$(iv) X_4(t) X_3(t+1) \leq 7000, \text{ for } t = 1, 2, \dots, 19$$

where $P_i(t)$ for all t and i are exogenous (prices), $X_i(t)$ for all t and i are endogenous (quantities), $\bar{X}(t)$ for all t are exogenous (CES-aggregated quantity), and A_i for all i , and r are parameters.

The Kuhn-Tucker optimality conditions for this problem are:

$$\begin{aligned} P_1(t) - P(t) \left[A_1 \bar{X}(t) / X_1(t) \right]^{1+r} + l(t) &= 0, \quad (t=1) \\ P_1(t) - P(t) \left[A_1 \bar{X}(t) / X_1(t) \right]^{1+r} + l(t) + l(t-1) &= 0, \quad (1 < t < 20) \\ P_1(t) - P(t) \left[A_1 \bar{X}(t) / X_1(t) \right]^{1+r} + l(t-1) &= 0, \quad (t=20) \end{aligned} \quad (25)$$

$$P_2(t) - P(t) \left[A_2 \bar{X}(t) / X_2(t) \right]^{1+r} - k(t) = 0, \quad \text{for all } t, \quad (26)$$

$$\begin{aligned} P_3(t) - P(t) \left[A_3 \bar{X}(t) / X_3(t) \right]^{1+r} + h(t-1) X_4(t-1) &= 0, \quad (t > 1) \\ P_3(t) - P(t) \left[A_3 \bar{X}(t) / X_3(t) \right]^{1+r} &= 0, \quad (t=1) \end{aligned} \quad (27)$$

$$\begin{aligned} P_4(t) - P(t) \left[A_4 \bar{X}(t) / X_4(t) \right]^{1+r} + h(t) X_3(t+1) &= 0, \quad (1 \leq t < 20) \\ P_4(t) - P(t) \left[A_4 \bar{X}(t) / X_4(t) \right]^{1+r} &= 0, \quad (t=20) \end{aligned} \quad (28)$$

$$P_5(t) - P(t) \left[A_5 \bar{X}(t) / X_5(t) \right]^{1+r} = 0, \quad \text{for all } t \quad (29)$$

$$\bar{X}(t) - \left[\sum_{i=1}^5 A_i^{1+r} X_i(t)^{-r} \right]^{-1/r} = 0, \quad \text{for all } t \quad (30)$$

$$l(t) [X_1(t) + X_1(t+1) - 500] = 0, \quad \text{for } t = 1, 2, \dots, 19 \quad (31)$$

$$k(t) [50 - X_2(t)] = 0, \quad \text{for all } t \quad (32)$$

$$h(t) [X_4(t) X_3(t+1) - 7000] = 0, \quad \text{for } t = 1, 2, \dots, 19 \quad (33)$$

$$X_1(t) + X_1(t+1) \leq 500, \quad \text{for } t = 1, 2, \dots, 19 \quad (34)$$

$$50 - X_2(t) \leq 0, \quad \text{for all } t \quad (35)$$

$$X_4(t) X_3(t+1) \leq 7000, \quad \text{for } t = 1, 2, \dots, 19 \quad (36)$$

$$l(t) \geq 0, \quad \text{for } t = 1, 2, \dots, 19 \quad (37)$$

$$k(t) \geq 0, \text{ for all } t \quad (38)$$

$$h(t) \geq 0, \text{ for } t = 1, 2, \dots, 19 \quad (39)$$

where $l(t), k(t)$ and $h(t)$ are the shadow prices of corresponding inequality constraints, respectively; and $P(t)$ are the shadow prices of the equality constraints. All the shadow prices are endogenous variables.

To be able to use GEMPACK, at the first step an initial solution satisfying all of the above equations is required. Discussions on model calibration or how an initial solution could be obtained are beyond the scope of this paper. Now, it is assumed that an initial solution for the above equations is found, which is shown in Table 1. One may check that the initial solution does satisfy all equations (25)-(39).

Table 1. Parameter values and initial data for the CES example for all t

i	A_i	$P_i^0(t)$	$X_i^0(t)$
1	0.074073271	1	120
2	0.197528823	2	80
3	0.277775297	3	50
4	0.296294643	4	30
5	0.154323867	5	10
$\bar{X}^0(t)$	222.22	$P^0(t)$	2.7
$l^0(t)$	0	$k^0(t)$	0
$h^0(t)$	0	r	-0.5

Introducing perturbations as well as converting the inequalities to equalities, the equations (31)-(39) become:

$$[l(t) + e][e + 500 - X_1(t) - X_1(t+1)] + e^0[e^0 + 500 - X_1^0(t) - X_1^0(t+1) + l^0(t)]q = 0, \text{ for } t = 1, 2, \dots, 19 \quad (40)$$

$$[k(t) + e][e + X_2(t) - 50] + e^0[e^0 + X_2^0(t) - 50 + k^0(t)]q = 0, \text{ for all } t \quad (41)$$

$$[h(t) + e][e + 7000 - X_4(t)X_3(t+1)] + e^0[e^0 + 7000 - X_4^0(t)X_3^0(t+1) + h^0(t)]q = 0, \text{ for } t = 1, 2, \dots, 19 \quad (42)$$

$$500 - X_1(t) - X_1(t+1) + e = [Z_1(t)]^2, \text{ for } t = 1, 2, \dots, 19 \quad (43)$$

$$X_2(t) - 50 + e = [Z_2(t)]^2, \text{ for all } t \quad (44)$$

$$7000 - X_4(t)X_3(t+1) + e = [Z_3(t)]^2, \text{ for } t = 1, 2, \dots, 19 \quad (45)$$

$$l(t) + e = [V_1(t)]^2, \text{ for } t = 1, 2, \dots, 19 \quad (46)$$

$$k(t) + e = [V_2(t)]^2, \text{ for all } t \quad (47)$$

$$h(t) + e = [V_3(t)]^2, \text{ for } t = 1, 2, \dots, 19 \quad (48)$$

Now, for the perturbed system given by equations (25)-(30) and (40)-(48), initialisation on all variables existing in the original equations (25)-(39) is the same as before, i.e., as shown in Table 1; q is initialised at -1 (i.e., $q^0 = -1$); e is initialised arbitrarily at 0.01 (i.e., $e^0 = 0.01$); and $Z_i(t)$ and $V_i(t)$ for $i = 1, 2, 3$ and for each t are initialised by the following formulae:

$$Z_1^0(t) = \sqrt{500 - X_1^0(t) - X_1^0(t+1) + e^0}, \text{ for } t = 1, 2, \dots, 19$$

$$Z_2^0(t) = \sqrt{X_2^0(t) - 50 + e^0}, \text{ for all } t$$

$$Z_3^0(t) = \sqrt{7000 - X_4^0(t)X_3^0(t+1) + e^0}, \text{ for } t = 1, 2, \dots, 19$$

$$V_1^0(t) = \sqrt{l^0(t) + e^0}, \text{ for } t = 1, 2, \dots, 19$$

$$V_2^0(t) = \sqrt{k^0(t) + e^0}, \text{ for all } t$$

$$V_3^0(t) = \sqrt{h^0(t) + e^0}, \text{ for } t = 1, 2, \dots, 19.$$

The solution for the original equations (25)-(39) can be obtained by solving the perturbed system given by equations (25)-(30) and (40)-(48), by shocking e and q to zero using GEMPACK for given paths of all exogenous variables in the unperturbed system.

Now, for example, let the aggregated quantity (or demand) $\bar{X}(t+1) = 1.06\bar{X}(t)$, for $t = 1, \dots, 19$, i.e., 6% increase per time step, with $\bar{X}(1) = \bar{X}^0(1) = 222.22$ as shown in Table 1; let the price (of the input 2) $P_2(t+1) = 1.1P_2(t)$, for $t = 1, \dots, 19$, i.e., 10% increase per time step, with $P_2(1) = P_2^0(1) = 2$ as shown in Table 1; and let all other exogenous variables remain the same values as shown in Table 1 for all t . For these given paths of exogenous variables, the solution for the original equations (25)-(39) can be obtained by solving the perturbed system (i.e., equations (25)-(30) and (40)-(48)) by shocking e and q to zero.

As $P_2(t)$ goes up, demand for the input 2, i.e., $X_2(t)$ will go down. It is expected that the lower bound constraint on $X_2(t)$ will bind. Similarly, as the aggregated demand $\bar{X}(t)$ goes up, the intertemporal upper bound constraints on demand for individual inputs $X_1(t) + X_1(t+1)$ and $X_4(t)X_3(t+1)$ will also bind. The solution for the problem obtained by GEMPACK is shown in Figures 2-4, where it can be seen that the inequality constraint on $X_1(t) + X_1(t+1)$ starts to bind and its associated shadow price becomes positive at $t = 9$ (Figure 2); similar results on the other two inequality constraints are shown in Figures 3 and 4. It is interesting to note that the fluctuated shadow price $l(t)$ (Figure 2) and the little hump on $X_2(t)$ around $t = 13$ (Figure 3) are caused by the intertemporal effect. The same solution was also obtained by solving the optimisation problem using GAMS and Excel. It is also further checked that the full original Kuhn-Tucker optimality conditions given by equations (25)-(39) are satisfied with the solution obtained by GEMPACK. This clearly demonstrates the effectiveness of the proposed method in solving complementarity conditions using GEMPACK.

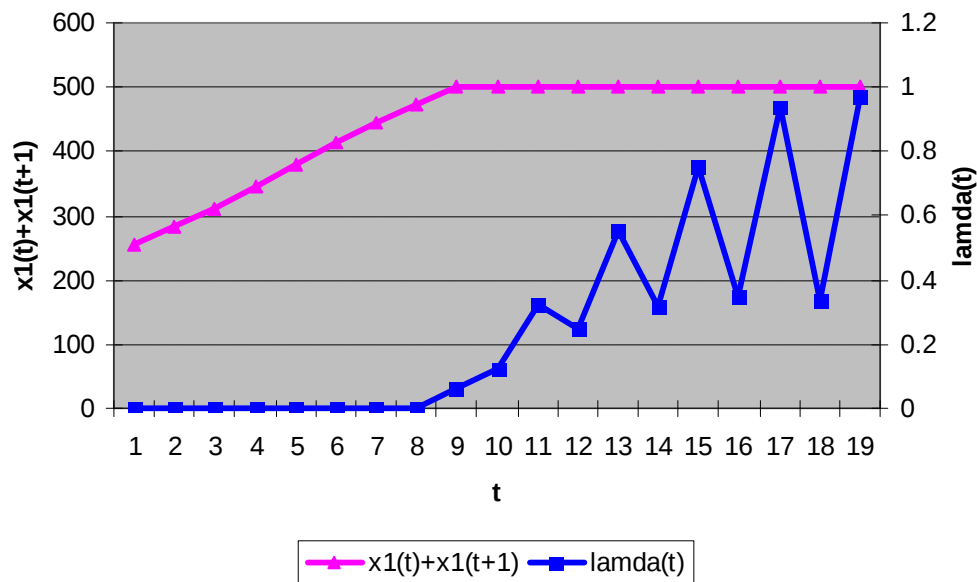


Figure 2. GEMPACK result on the CES example showing the inequality constraint on $X_1(t) + X_1(t+1)$ binding and its associated shadow price $l(t)$, with given shocks on $\bar{X}(t)$ and $P_2(t)$.

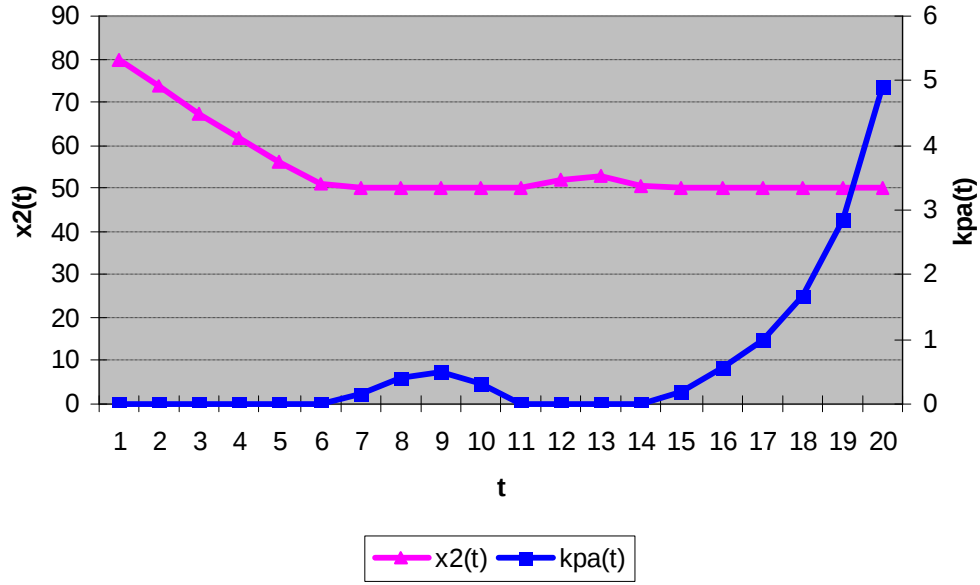


Figure 3. GEMPACK result on the CES example showing the inequality constraint on $X_2(t)$ binding and its associated shadow price $k(t)$, with given shocks on $\bar{X}(t)$ and $P_2(t)$, noting the little hump on $X_2(t)$ around $t = 13$ due to the intertemporal effect.

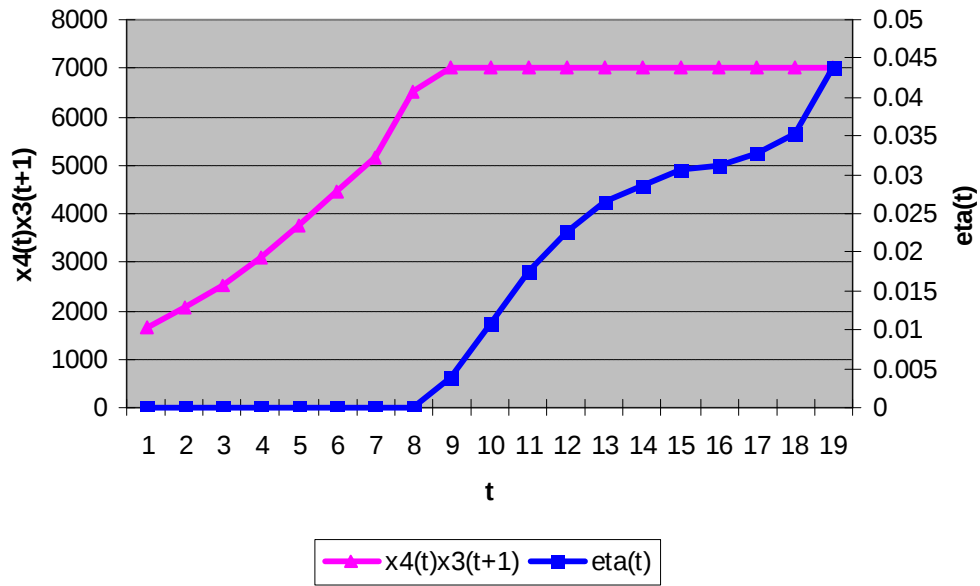


Figure 4. GEMPACK result on the CES example showing the inequality constraint on $X_4(t)X_3(t+1)$ binding and its associated shadow price $h(t)$, with given shocks on $\bar{X}(t)$ and $P_2(t)$.

Harrison *et al.* (2002)'s two pass method was also applied to this example, but it did not work, even with very large number of sub-intervals. The error message was "E-problem with accurate simulation".

3.4. Example of perfect substitution

This example is a typical one for testing the proposed method in solving corner solution problems. The ability for GEMPACK to solve this type of problems would extend GEMPACK-based modelling freedoms by allowing perfect substitution cases in the models if necessary.

Consider the following perfect substitution example:

$$\min_{X(t), t=1, \dots, 20} \sum_{t=1}^{20} \sum_{i=1}^3 P_i(t) X_i(t)$$

subject to: for each t ,

$$(i) \quad \bar{X}(t) = \sum_{i=1}^3 X_i(t)$$

$$(ii) \quad X_i(t) \leq 100, \text{ for } i = 1, 2$$

$$(iii) \quad X_i(t) \geq 0, \text{ for } i = 1, 2, 3$$

where $P_i(t)$ for all t and all i are exogenous (prices), $X_i(t)$ for all t and all i are endogenous (quantities), and $\bar{X}(t)$ for all t are exogenous (total quantity). $X_i(t)$ for all i are perfectly substitutable because of the condition (i).

For each t , the Kuhn-Tucker optimality conditions for this problem are:

$$P_i(t) - P(t) + l_i(t) - k_i(t) = 0, \text{ for } i = 1, 2 \quad (49)$$

$$P_i(t) - P(t) - k_i(t) = 0, \text{ for } i = 3$$

$$\bar{X}(t) = \sum_{i=1}^3 X_i(t) \quad (50)$$

$$l_i(t)[X_i(t) - 100] = 0, \text{ for } i = 1, 2 \quad (51)$$

$$k_i(t)X_i(t) = 0, \text{ for } i = 1, 2, 3 \quad (52)$$

$$X_i(t) \leq 100, \text{ for } i = 1, 2 \quad (53)$$

$$X_i(t) \geq 0, \text{ for } i = 1, 2, 3 \quad (54)$$

$$l_i(t) \geq 0, \text{ for } i = 1, 2, \text{ and} \quad (55)$$

$$k_i(t) \geq 0, \text{ for } i = 1, 2, 3 \quad (56)$$

where $l_i(t)$ and $k_i(t)$ are the shadow prices of corresponding inequality constraints, respectively; and $P(t)$ are the shadow prices of the equality constraints. All the shadow prices are endogenous.

Table 2. Initial data for the perfect substitution example for all t

i	$X_i^0(t)$	$P_i^0(t)$	$l_i^0(t)$	$k_i^0(t)$
1	100	1	1	0
2	10	2	0	0
3	0	3	NA	1
$\overline{X}^0(t)$ 110			$P^0(t)$ 2	

An initial solution for this problem is given in Table 2, noting that the upper bound for $X_1(t)$ and the lower bound for $X_3(t)$ are binding initially (hence they have corresponding positive shadow prices as shown in Table 2). Introducing perturbations as well as converting the inequalities to equalities, the equations (51)-(56) become:

$$[l_i(t)+e][100-X_i(t)+e]+e^0[e^0-X_i^0(t)+100+l_i^0(t)]q=0, \text{ for } i=1, 2 \quad (57)$$

$$[k_i(t)+e][X_i(t)+e]+e^0[e^0+X_i^0(t)+k_i^0(t)]q=0, \text{ for } i=1, 2, 3 \quad (58)$$

$$100-X_i(t)+e=[Z_i(t)]^2, \text{ for } i=1, 2 \quad (59)$$

$$X_i(t)+e=[U_i(t)]^2, \text{ for } i=1, 2, 3 \quad (60)$$

$$l_i(t)+e=[V_i(t)]^2, \text{ for } i=1, 2, \text{ and} \quad (61)$$

$$k_i(t)+e=[W_i(t)]^2, \text{ for } i=1, 2, 3 \quad (62)$$

Now, for the perturbed system given by equations (49)-(50) and (57)-(62), initialisation on all variables existing in the original equations (49)-(56) is the same as before, i.e., as shown in Table 2; q is initialised at -1 (i.e., $q^0 = -1$); e is initialised arbitrarily at 1 (i.e., $e^0 = 1$); and all remaining variables are initialised by the following formulae:

$$Z_i^0(t) = \sqrt{100 - X_i^0(t) + e^0}, \text{ for } i=1, 2$$

$$U_i^0(t) = \sqrt{X_i^0(t) + e^0}, \text{ for } i=1, 2, 3$$

$$V_i^0(t) = \sqrt{l_i^0(t) + e^0}, \text{ for } i=1, 2, \text{ and}$$

$$W_i^0(t) = \sqrt{k_i^0(t) + e^0}, \text{ for } i=1, 2, 3.$$

The solution for the original equations (49)-(56) can be obtained by solving the perturbed system given by equations (49)-(50) and (57)-(62), by shocking e and q to

zero for given paths of all exogenous variables in the original system using GEMPACK.

Now, for example, let the aggregated quantity (or demand) $\bar{X}(t+1) = 1.05\bar{X}(t)$, for $t = 1, \dots, 19$, i.e., 5% increase per time step, with $\bar{X}(1) = \bar{X}^0(1) = 110$ as shown in Table 2; let the price (of the input 1) $P_1(t+1) = 1.07P_1(t)$, for $t = 1, \dots, 19$, i.e., 7% increase per time step, with $P_1(1) = P_1^0(1) = 1$ as shown in Table 2; and let all other exogenous variables remain the same values as shown in Table 2 for all t . For these given paths of exogenous variables, the solution for the original equations (49)-(56) can be obtained by solving the perturbed system (i.e., equations (49)-(50) and (57)-(62)) by shocking e and q to zero.

As $P_1(t)$ goes up, it is expected that $X_1(t)$ moves down from its upper bound (initially binding) toward its lower bound at zero (will be binding when its price becomes higher than all others'). How fast it moves away from its upper to its lower bound depends on how quick its price exceeds the prices of others. Similarly, as $\bar{X}(t)$ goes up, $X_2(t)$ will hit its capacity even if $P_1(t)$ holds constant – now because $P_1(t)$ goes up as well, this further speeds up $X_2(t)$ to hit its capacity.

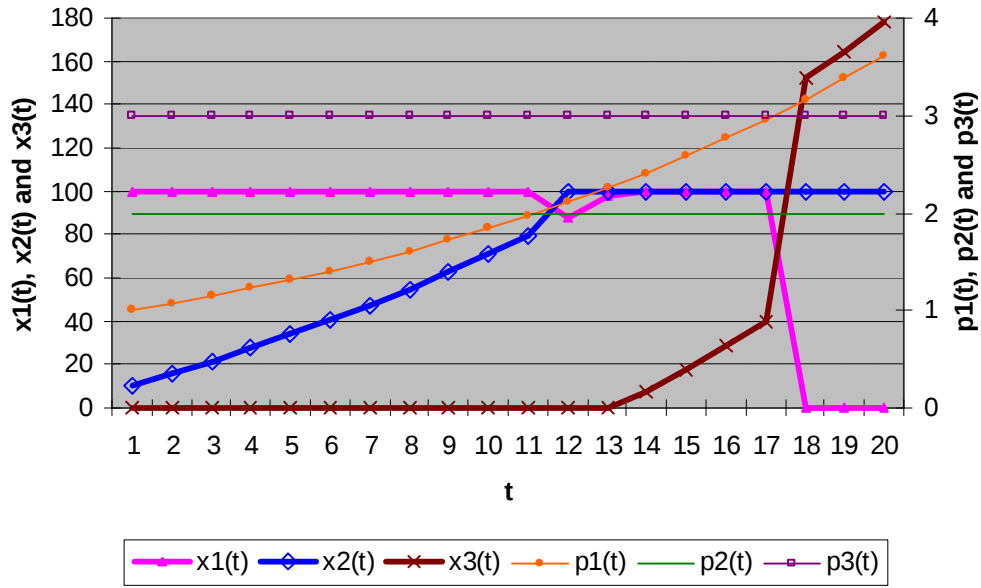


Figure 5. GEMPACK's solution with shocks on $\bar{X}(t)$ and $P_1(t)$ in the perfect substitution example, showing the upper and lower bounds binding on $X_i(t)$, which is caused by the movement of individual prices (see the price curves) and total demand/quantity change. Note that only lower bound is imposed for $X_3(t)$.

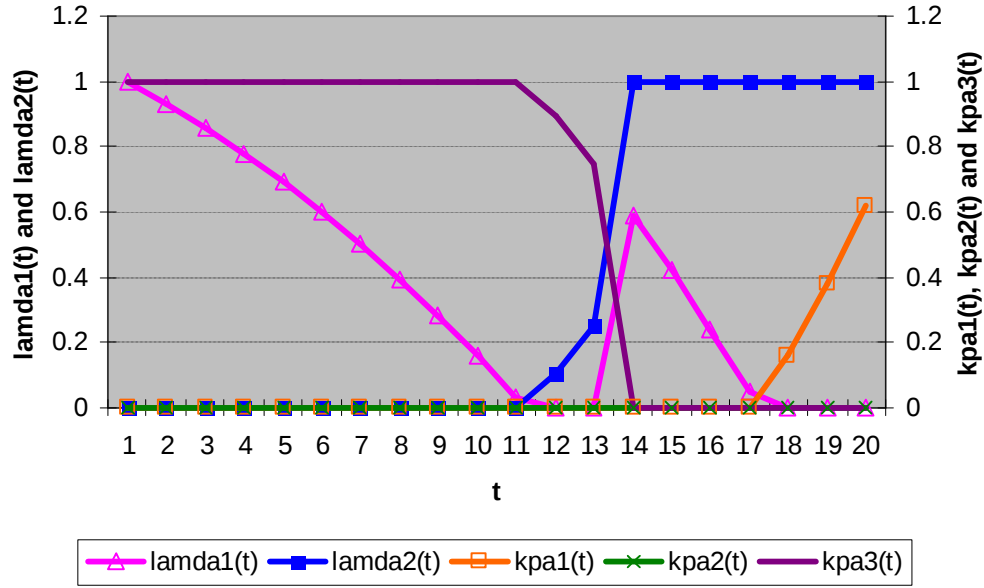


Figure 6. GEMPACK's solution with shocks on $\bar{X}(t)$ and $P_1(t)$ in the perfect substitution example, showing the shadow price movements associated with the lower and upper bound constraints on $X_i(t)$ (this is the dual result of Figure 5). Note that the lower bound for $X_2(t)$ never binds, so its shadow price $k_2(t)$ stays at zero all the time.

The solution for the problem obtained by GEMPACK is shown in Figures 5-6, where the expected result discussed above is clearly shown in the charts. Shadow price movements (Figure 6) match their corresponding primal variables' movements (Figure 5). The dip on the curve of $X_1(t)$ at $t = 12$ is due to that $P_1(t)$ exceeds $P_2(t)$ but is still below $P_3(t)$ – in order to meet the total output or demand requirement, at this time, it is economic to use $X_1(t)$ instead of $X_3(t)$ when $X_2(t)$ hits its capacity constraint. The same reason is for that $X_1(t)$ moves back to hit its capacity at $t = 14$. When $P_1(t)$ exceeds $P_3(t)$ at $t = 18$, $X_1(t)$ hits its lower bound at zero and $X_3(t)$ jumps higher. It is also checked that the full original Kuhn-Tucker optimality conditions given by equations (49)-(56) are satisfied with the solution obtained by GEMPACK.

Harrison *et al.* (2002)'s two pass method was applied to this example, but it did not work either, even with very large number of sub-intervals. The error message was "LHS structurally singular" (on those complementarity equations).

3.5. Example of a simple intertemporal general equilibrium model

Having tested a few examples above, the example provided in this section is slightly complicated, which is a simple general equilibrium model and has intertemporal dynamics. The reason for providing this example is to demonstrate the effectiveness of the proposed method applied to a framework of intertemporal general equilibrium.

Consider the following hypothetical intertemporal general equilibrium model:

- $$\min_{X(t), t=1, \dots, 20} -\sum_{t=1}^{20} a(t) \ln[C(t)]$$
 subject to: for each t ,
- (a) production by two technologies: $Y_i(t) = A_i(t)K_i(t)^{g_i} \{[L_i(t)+1]^{1-g_i} - 1\}$, for $i = 1, 2$
 - (b) total output: $\bar{Y}(t) = \sum_{i=1}^2 Y_i(t)$
 - (c) capital cumulation: $K_i(t+1) = [1-d]K_i(t) + I_i(t)$, for $i = 1, 2$
 - (d) terminal condition on investment for technology 1: $I_1(20) = [d+g]K_1(20)$
 - (e) terminal condition on investment for technology 2: $I_2(20) = I_2(19)$
 - (f) output equilibrium condition: $C(t) + \sum_{i=1}^2 I_i(t) = \bar{Y}(t)$
 - (g) labour equilibrium condition: $\sum_{i=1}^2 L_i(t) = \bar{L}(t)$
 - (h) constraint on output by technology 2: $Y_2(t) \leq m(t)\bar{Y}(t)$
 - (i) non-negative investment: $I_i(t) \geq 0$
 - (j) non-negative labour: $L_i(t) \geq 0$

where at each t , $C(t)$ is the consumption, $Y_i(t)$ is the production by technology i , $K_i(t)$ is the capital used by technology i , $L_i(t)$ is the labour used by technology i , $A_i(t)$ is the productivity of technology i , $\bar{Y}(t)$ is the total production, $I_i(t)$ is the investment for technology i , $\bar{L}(t)$ is the total labour, $m(t)$ is the maximum share allowed of output by technology 2 (imagine this is dirty technology), d is the capital depreciation rate, g is the growth rate, g_i is the share parameter in the production function for technology i , and $a(t)$ is the weight coefficient in the objective function defined by

$$a(t) = \{[1+g]/[1+r]\}^{t-1}, \text{ for } t < 20, \text{ and}$$

$$a(t) = \{[1+g]/[1+r]\}^{t-1} / \{1 - [1+g]/[1+r]\} \text{ for } t = 20,$$

where r is the discount rate.

Among the list, d , g , g_i for all i and r are parameters; $\bar{L}(t)$, $m(t)$, $A_i(t)$ and $K_i(1)$ (capital at the base year) for all i are exogenous; and all others are endogenous variables to minimize the negative intertemporal utility: $-\sum_{t=1}^{20} a(t) \ln[C(t)]$. This form of intertemporal utility is taken from the GAMS code by Lau *et al.* (2000) (see also Barr and Manne (1967)).

For each t , the Kuhn-Tucker optimality conditions for this problem are

$$PEY(t)C(t) = a(t)[1+r]^{t-1}, \quad (63)$$

$$\begin{aligned} PY_1(t) &= PTY(t), \\ PY_2(t) &= PTY(t) - PDY(t), \end{aligned} \quad (64)$$

$$PTY(t) = PEY(t) + m(t)PDY(t), \quad (65)$$

$$PL_i(t) = PTL(t) - PY_i(t)[1 - g_i][Y_i(t) + A_i(t)K_i(t)^{g_i}] / [L_i(t) + 1], \text{ for } i = 1, 2 \quad (66)$$

$$PK_i(t) = [1 - d]PK_i(t+1) / [1 + r] + PY_i(t)g_i Y_i(t) / K_i(t), \text{ for } i = 1, 2 \text{ and } t < 20 \quad (67)$$

$$PK_1(t) = PY_1(t)g_1 Y_1(t) / K_1(t) - [d + g]PKT_1, \text{ for } t = 20 \quad (68)$$

$$PK_2(t) = PY_2(t)g_2 Y_2(t) / K_2(t), \text{ for } t = 20 \quad (69)$$

$$PI_i(t) = PK_i(t+1) / [1 + r] - PEY(t), \text{ for } i = 1 \text{ and } t < 20; i = 2 \text{ and } t < 19 \quad (70)$$

$$PI_2(t) = PK_2(t+1) / [1 + r] - PEY(t) - PKT_2 / [1 + r], \text{ for } i = 2 \text{ and } t = 19 \quad (71)$$

$$PI_i(t) = PKT_i - PEY(t), \text{ for } i = 1, 2 \text{ and } t = 20 \quad (72)$$

$$K_i(t+1) = [1 - d]K_i(t) + I_i(t), \text{ for } i = 1, 2 \quad (73)$$

$$I_1(20) = [d + g]K_1(20), \quad (74)$$

$$I_2(20) = I_2(19), \quad (75)$$

$$Y_i(t) = A_i(t)K_i(t)^{g_i} \{ [L_i(t) + 1]^{1-g_i} - 1 \}, \text{ for } i = 1, 2 \quad (76)$$

$$\bar{Y}(t) = \sum_{i=1}^2 Y_i(t), \quad (77)$$

$$C(t) + \sum_{i=1}^2 I_i(t) = \bar{Y}(t), \quad (78)$$

$$\sum_{i=1}^2 L_i(t) = \bar{L}(t), \quad (79)$$

$$PDY(t)[m(t)\bar{Y}(t) - Y_2(t)] = 0, \quad (80)$$

$$PI_i(t)I_i(t) = 0, \quad (81)$$

$$PL_i(t)L_i(t) = 0, \quad (82)$$

$$Y_2(t) \leq m(t)\bar{Y}(t), \quad (83)$$

$$I_i(t) \geq 0, \quad (84)$$

$$L_i(t) \geq 0, \quad (85)$$

$$PDY(t) \geq 0, \quad (86)$$

$$PI_i(t) \geq 0, \quad (87)$$

$$PL_i(t) \geq 0, \quad (88)$$

where $PK_i(t)$ is the shadow price of equation (73), PKT_1 is the shadow price of equation (74), PKT_2 is the shadow price of equation (75), $PY_i(t)$ is the shadow price

of equation (76), $PTY(t)$ is the shadow price of equation (77), $PEY(t)$ is the shadow price of equation (78), $PTL(t)$ is the shadow price of equation (79), $PDY(t)$ is the shadow price of equation (83), $PI_i(t)$ is the shadow price of equation (84), and $PL_i(t)$ is the shadow price of equation (85). All the shadow prices are endogenous.

An initial solution and parameter values for the above equations (63)-(88) is given in Table 3. In addition, $\bar{Y}^0(t) = \sum_{i=1}^2 Y_i^0(t)$ and $\bar{L}^0(t) = \sum_{i=1}^2 L_i^0(t)$. As there are dynamic equations (e.g., capital cumulation equation (73)) as well as terminal conditions, the specified initial solution shown in Table 3 does not satisfy these equations. A usual GEMPACK homotopic approach is used to deal with these equations, which are given below:

$$PK_i(t) = [1-d]PK_i(t+1)/[1+r] + PY_i(t)g_iY_i(t)/K_i(t) + \{[1-d]PK_i^0(t+1)/[1+r] + PY_i^0(t)g_iY_i^0(t)/K_i^0(t) - PK_i^0(t)\}q, \text{ for all } i \text{ and } t < 20 \quad (89)$$

$$PK_1(t) = PY_1(t)g_1Y_1(t)/K_1(t) - [d+g]PKT_1 + \{PY_1^0(t)g_1Y_1^0(t)/K_1^0(t) - [d+g]PKT_1^0 - PK_1^0(t)\}q, \text{ for } t = 20 \quad (90)$$

$$PK_2(t) = PY_2(t)g_2Y_2(t)/K_2(t) + \{PY_2^0(t)g_2Y_2^0(t)/K_2^0(t) - PK_2^0(t)\}q, \text{ for } t = 20 \quad (91)$$

$$PI_i(t) = PK_i(t+1)/[1+r] - PEY(t) + \{PK_i^0(t+1)/[1+r] - PEY^0(t) - PI_i^0(t)\}q, \text{ for } i = 1 \text{ and } t < 20; i = 2 \text{ and } t < 19 \quad (92)$$

$$PI_2(t) = PK_2(t+1)/[1+r] - PEY(t) - PKT_2/[1+r] + \{PK_2^0(t+1)/[1+r] - PEY^0(t) - PKT_2^0/[1+r] - PI_2^0(t)\}q, \text{ for } i = 2 \text{ and } t = 19 \quad (93)$$

$$PI_i(t) = PKT_i - PEY(t) + \{PKT_i^0 - PEY^0(t) - PI_i^0(t)\}q, \text{ for } i = 1, 2 \text{ and } t = 20 \quad (94)$$

$$K_i(t+1) = [1-d]K_i(t) + I_i(t) + \{[1-d]K_i^0(t) + I_i^0(t) - K_i^0(t+1)\}q, \text{ for } i = 1, 2 \quad (95)$$

$$I_1(20) = [d+g]K_1(20) + \{[d+g]K_1^0(20) - I_1^0(20)\}q, \quad (96)$$

where q is as defined before, which is exogenous, takes an initial value of -1 and will be shocked to zero.

Table 3. Parameter values and initial data for the general equilibrium example for all t

i	$A_i^0(t)$	$L_i^0(t)$	g_i	$K_i^0(t)$	$Y_i^0(t)$	$I_i^0(t)$
1	1	0	0.6	1	0	0
2	3	10	0.5	10	21.97743	18.11128

i	$PY_i^0(t)$	$PK_i^0(t)$	PKT_i^0	$PI_i^0(t)$	$PL_i^0(t)$
1	0.258655	-0.01179	0	0.270938	0.266465
2	0.258655	0.532538	0	0	0

g	0.06	$C^0(t)$	3.866152	$PDY(t)$	0
r	0.07	$PTY(t)$	0.258655	$PTL(t)$	0.369927
d	0.04	$PEY(t)$	0.258655	$m(t)$	1.01

So, the equations that need to be solved by GEMPACK are: equations (63)-(66) and (75)-(96). One may check that the initial solution given in Table 3 satisfies all of these equations, noting that the homotopic variable q will be shocked to zero.

Introducing perturbations as well as converting the inequalities to equalities, the equations (80)-(88) become: for each t ,

$$[PDY(t) + e][m(t)\bar{Y}(t) - Y_2(t) + e] + e^0[e^0 + PDY^0(t) + m^0(t)\bar{Y}^0(t) - Y_2^0(t)]q = 0, \quad (97)$$

$$[PI_i(t) + e][I_i(t) + e] + e^0[e^0 + PI_i^0(t) + I_i^0(t)]q = 0, \quad (98)$$

$$[PL_i(t) + e][L_i(t) + e] + e^0[e^0 + PL_i^0(t) + L_i^0(t)]q = 0, \quad (99)$$

$$m(t)\bar{Y}(t) - Y_2(t) + e = [ZY(t)]^2, \quad (100)$$

$$I_i(t) + e = [ZI_i(t)]^2, \quad (101)$$

$$L_i(t) + e = [ZL_i(t)]^2, \quad (102)$$

$$PDY(t) + e = [VY(t)]^2, \quad (103)$$

$$PI_i(t) + e = [VI_i(t)]^2, \quad (104)$$

$$PL_i(t) + e = [VL_i(t)]^2. \quad (105)$$

Now, for the perturbed system given by equations (63)-(66), (75)-(79) and (89-105), initialisation on all variables existing in the original equations (63)-(88) is the same as before, i.e., as shown in Table 3; q is initialised at -1 as mentioned earlier; e is initialised arbitrarily at 0.4 (i.e., $e^0 = 0.4$); and all remaining variables are initialised by the following formulae:

$$ZY^0(t) = \sqrt{m^0(t)\bar{Y}^0(t) - Y_2^0(t) + e^0},$$

$$ZI_i^0(t) = \sqrt{I_i^0(t) + e^0},$$

$$ZL_i^0(t) = \sqrt{L_i^0(t) + e^0},$$

$$VY^0(t) = \sqrt{PDY^0(t) + e^0},$$

$$VI_i^0(t) = \sqrt{PI_i^0(t) + e^0},$$

$$VL_i^0(t) = \sqrt{PL_i^0(t) + e^0}.$$

The solution for the original equations (63)-(88) can be obtained by solving the perturbed system given by equations (63)-(66), (75)-(79) and (89)-(105), by shocking e and q to zero for given paths of all exogenous variables in the original equations using GEMPACK.

As there are equations (i.e. equations (89)-(96)) with homotopic variables, it should be noted that the initial solution given in Table 3 is not the true solution for the original system (i.e., equations (63)-(88)) that needs to be solved. The true solution for the original system given by equations (63)-(88) can be obtained by solving the equations (63)-(66), (75)-(79) and (89)-(105) jointly by shocking e and q to zero, together with shocks if any on all other exogenous variables.

First, let all exogenous variables in the original system keep the same values as those in Table 3 (i.e., without shocks on them), the results for this run which is called ‘base-run’ are provided in Figures 7-11 showing the time-paths of all inequality constraints and their associated shadow prices. All shadow price movements match the movements of the corresponding primal variables. The same result was also obtained by a NLP solver in GAMS.

Now, let some exogenous variables in the original system change their paths (i.e., with shocks on them) to see how the solution changes from the ‘base-run’ solution. For example, let the productivity increase but differentially on the two technologies, and let the maximum output share constraint on technology 2 tightened, specifically, let $A_1(t+1) = 1.05A_1(t)$, $A_2(t+1) = 1.005A_2(t)$ and $m(t+1) = m(t) - 0.05$, for $t = 1, 2, \dots, 19$, where $A_i(1)$ for all i and $m(1)$ are given in Table 3. With these given paths of exogenous variables, the solution for the original system again can be obtained by solving the perturbed system (i.e., equations (63)-(66), (75)-(79) and (89)-(105)) by shocking e and q to zero using GEMPACK. Let this run be labelled ‘policy-run’.

As the productivity for technology 1 is improving, together with the constraint on the output by technology 2, there will be a shift of investment from technology 2 to technology 1, as well as flow of labour from technology 2 to technology 1.

The results for the ‘policy-run’ obtained by GEMPACK are shown in Figures 12-16, where the expected result discussed above is clearly shown in the charts. All shadow price movements match their corresponding primal variables’ movements. The fluctuation of time path of each variable is caused by the specific terminal conditions as well as the intertemporal effects. Again, the same result was also obtained by a NLP solver in GAMS. Certainly, it is also checked that the full original Kuhn-Tucker optimality conditions given by equations (63)-(88) are satisfied with the solution obtained by GEMPACK. As it can be seen from the results, the proposed method has worked effectively in solving inequality constraints or complementarity conditions.

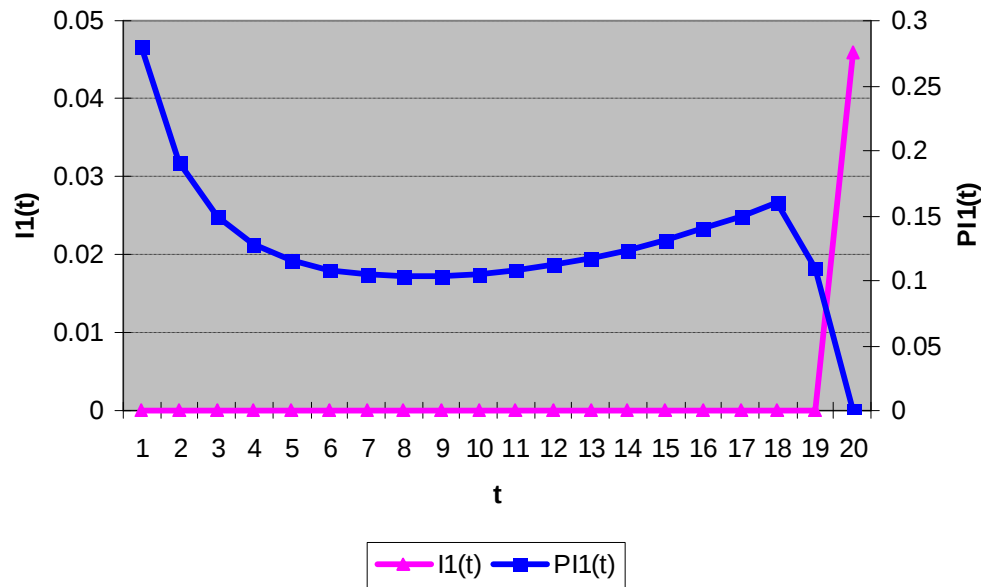


Figure 7. GEMPACK ‘base run’ results on the general equilibrium example showing the investment path and its associated shadow price for technology 1. The positive investment at the terminal year is caused by the terminal condition on investment (i.e., equation (74)).

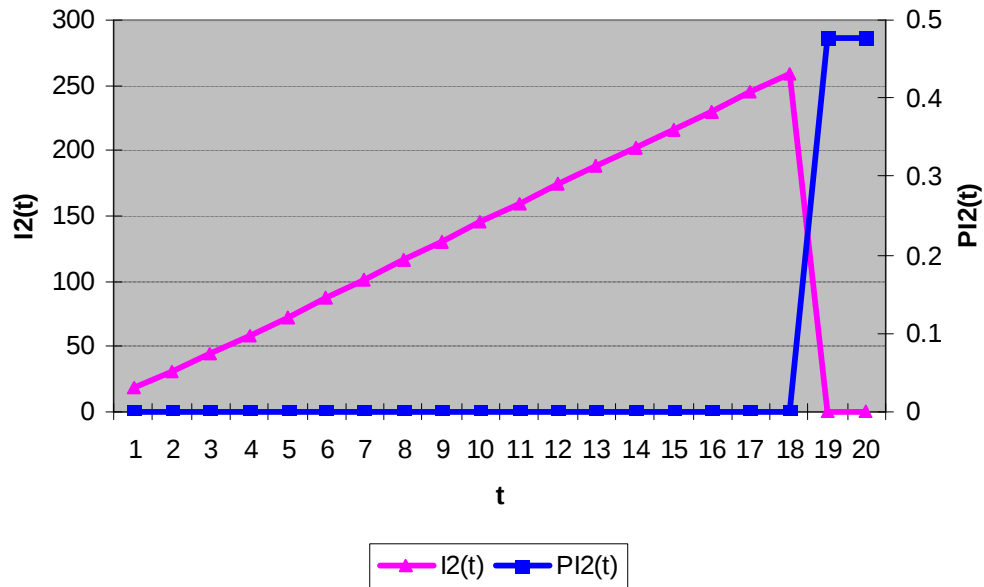


Figure 8. GEMPACK ‘base run’ results on the general equilibrium example showing the investment path and its associated shadow price for technology 2. The zero investment at the last two years is caused by the specified terminal condition on investment (i.e., equation (75)).

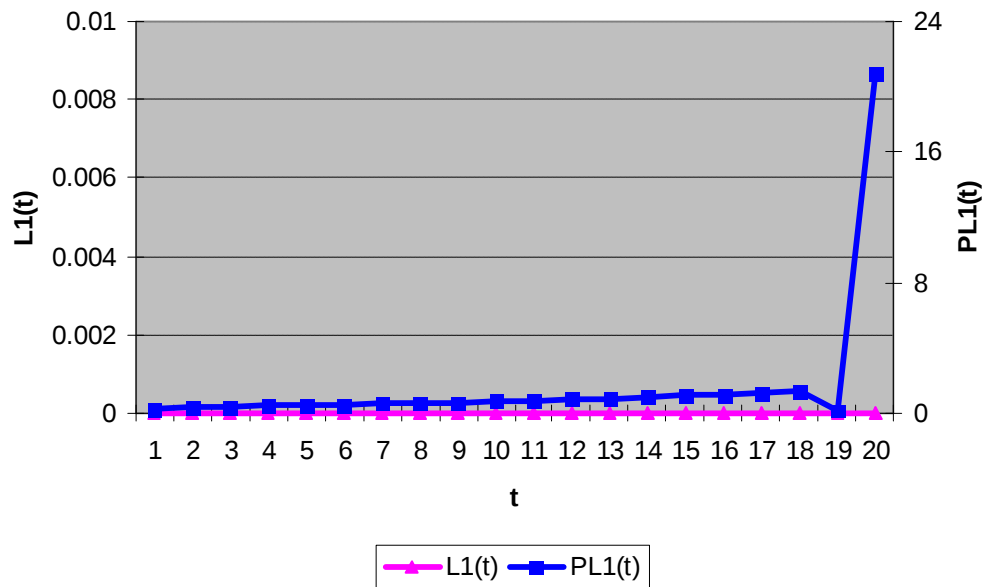


Figure 9. GEMPACK ‘base run’ results on the general equilibrium example showing the labour path and its associated shadow price for technology 1. Labour has not been used at all; hence, there is no production from technology 1. Note that the outputs by technology 1 and technology 2 are perfectly substitutable.

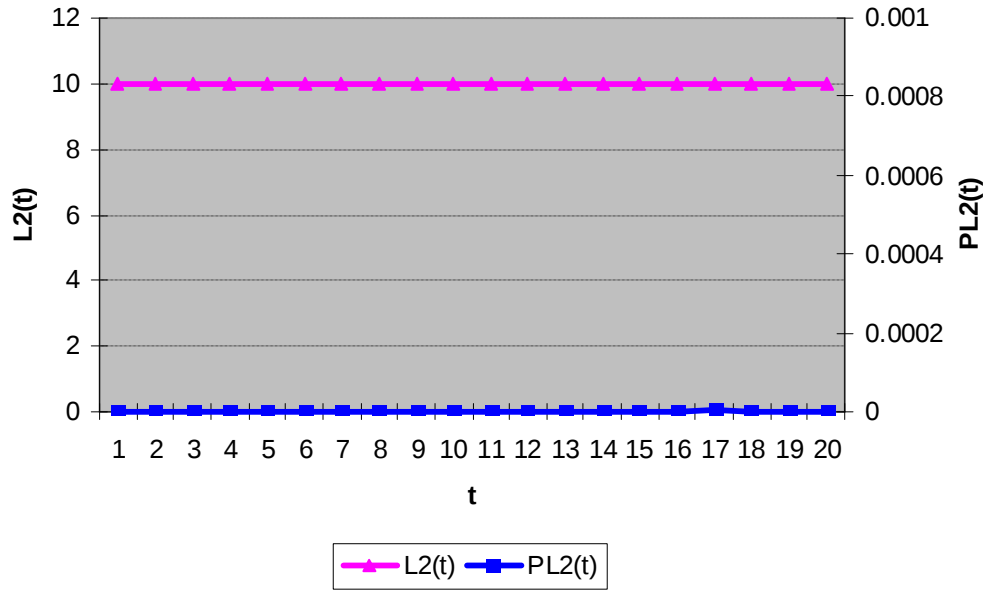


Figure 10. GEMPACK 'base run' results on the general equilibrium example showing the labour path and its associated shadow price for technology 2. All available labour is used here.

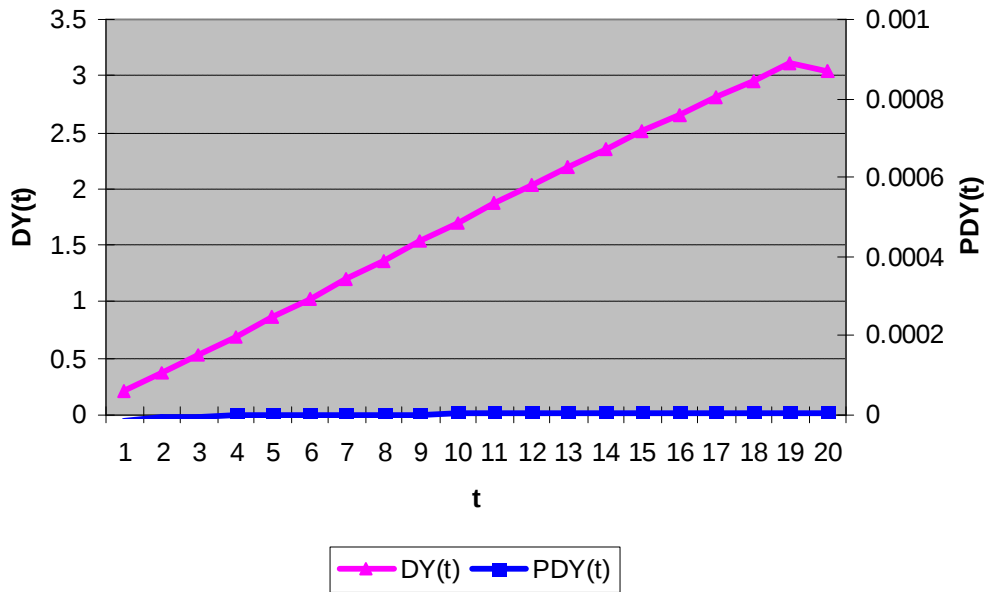


Figure 11. GEMPACK 'base run' results on the general equilibrium example showing the constraint on the output by technology 2 and its associated shadow price, where $DY(t) = m(t)\bar{Y}(t) - Y_2(t)$. Positive $DY(t)$ means the constraint is not binding, hence the shadow price is zero.

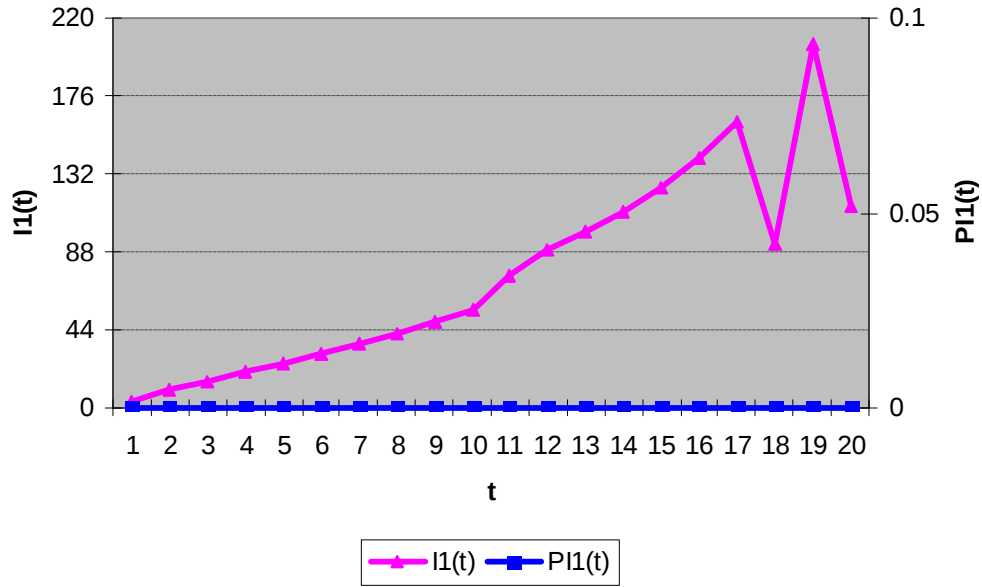


Figure 12. GEMPACK ‘policy run’ results on the general equilibrium example showing the investment path and its associated shadow price for technology 1, with shocks on $A_t(t)$ and $m(t)$. The fluctuation toward the end is caused by the terminal conditions and the intertemporal effects.

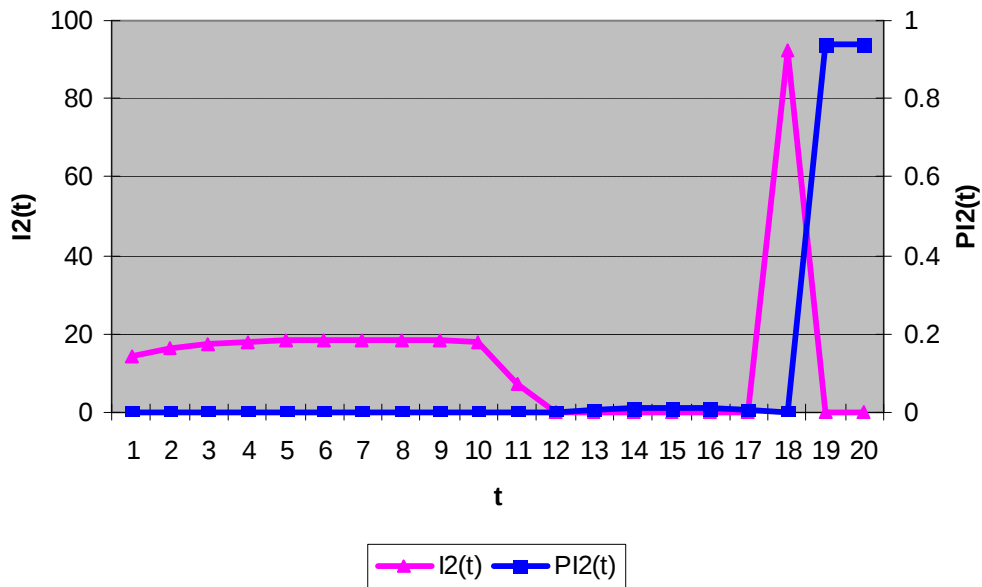


Figure 13. GEMPACK ‘policy run’ results on the general equilibrium example showing the investment path and its associated shadow price for technology 2, with shocks on $A_t(t)$ and $m(t)$. The fluctuation toward the end and the zero investment at the end are caused by the terminal conditions and the intertemporal effects.

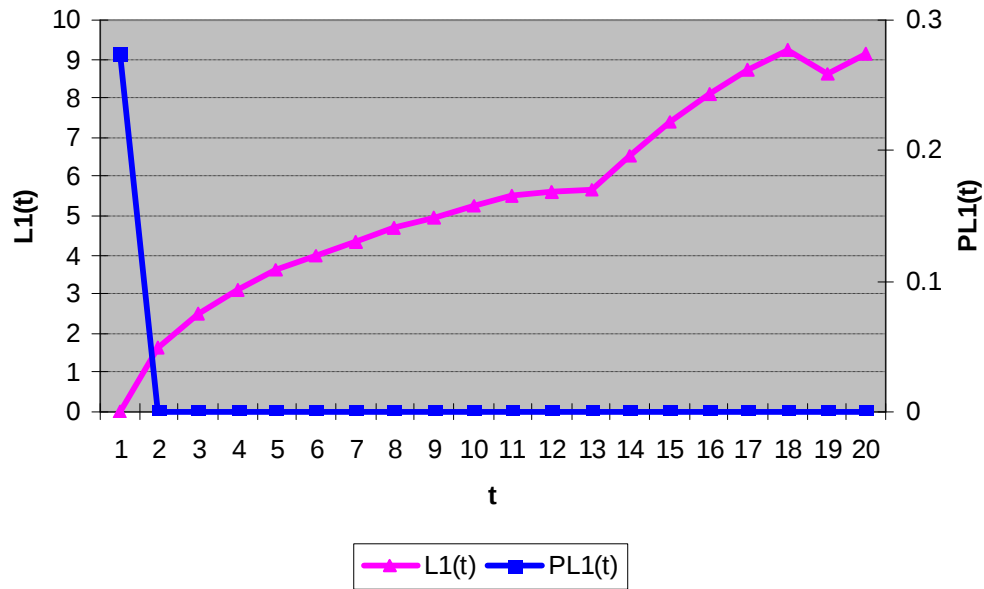


Figure 14. GEMPACK 'policy run' results on the general equilibrium example showing the labour path and its associated shadow price for technology 1, with shocks on $A_1(t)$ and $m(t)$.

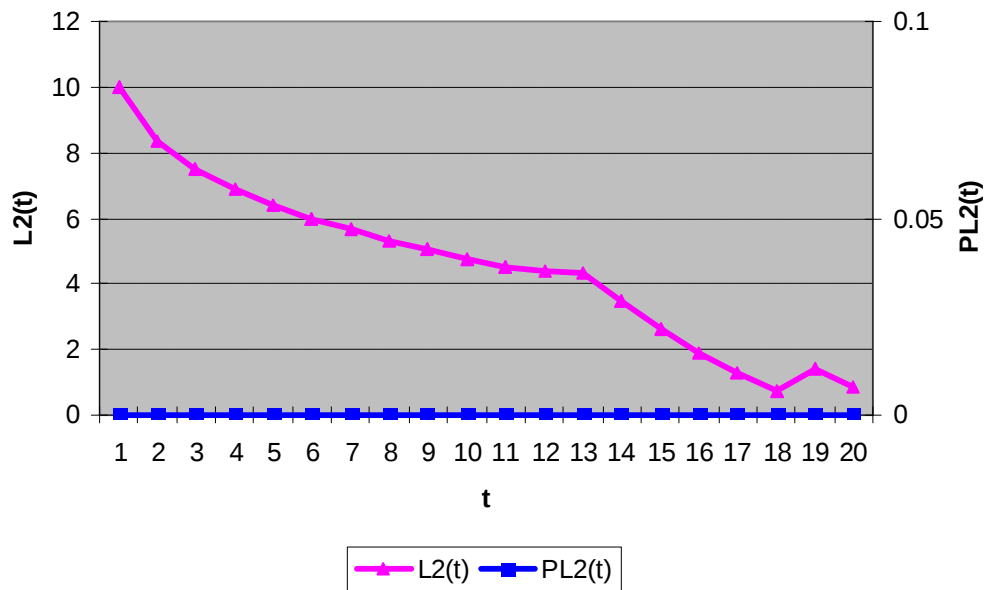


Figure 15. GEMPACK 'policy run' results on the general equilibrium example showing the labour path and its associated shadow price for technology 2, with shocks on $A_1(t)$ and $m(t)$.

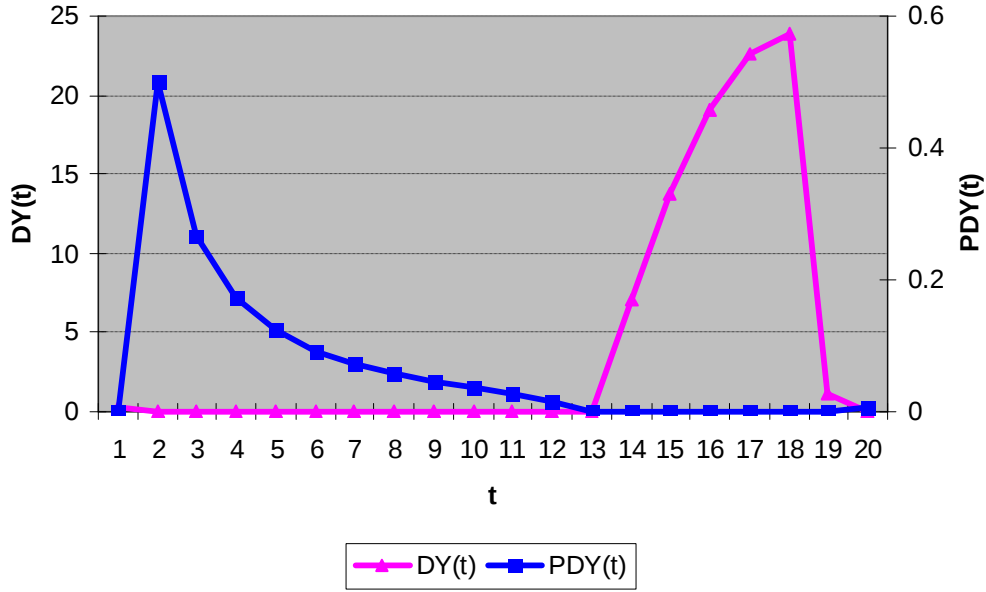


Figure 16. GEMPACK ‘policy run’ results on the general equilibrium example showing the constraint on the output by technology 2 and its associated shadow price, where $DY(t) = m(t)\bar{Y}(t) - Y_2(t)$.

As before, Harrison *et al.* (2002)’s two pass method was also applied to this example, but it did not work either on this example, even with very large number of sub-intervals. The error message was “LHS structurally singular”. The structurally singular problem is avoided using the method proposed in this paper – because of the perturbations applied – it helps GEMPACK to get away from the singularity (i.e. “zeros”).

3.6. Examples of existing models (GTEM, MMRF and TCGE)

The above examples are hypothetical examples. In this section, the proposed method is applied to three existing large computable general equilibrium models which are GTEM (Pant, 2007), MMRF (Adams *et al.*, 2003) and TCGE (Treasury Computable General Equilibrium model, Cao *et al.*, 2010). Both GTEM and MMRF are based on the Treasury versions of the models. The constraints imposed in the models below are hypothetical and are constructed only for testing the proposed method in solving equality constraints in the existing models.

3.6.1. Constraints on emission trading

In an emission permit trading scenario, if individual regions impose their own constraints on the number of permits which can be imported, then these constraints will attract additional prices on the permits within individual regions in addition to the world price of permits. This problem of constrained trading can be specified as follows:

Let $Quota(r)$ and $Emis(r)$ denote the emission quota allocated and the actual emissions of a region r . In the scenario without a constraint on the emission trading, a region can emit as much as it would like subject to its economic conditions and the world price of emission permits, and can buy permits ($= Emis(r) - Quota(r)$) from other regions. Hence, $Quota(r)$ does not really impose a cap on emissions for the

region r , rather just provides the rights or entitlements of permits that the region can trade with others. In the scenario with constraints on the emission trading, however, regional emissions can be capped. For testing the proposed method for solving inequalities, the following constraint is imposed in GTEM and MMRF:

$$Emis(r) \leq [1 + a(r)] \cdot Quota(r),$$

which is mathematically equivalent to

$$[Emis(r) - Quota(r)] / [a(r) \cdot Quota(r)] \leq 1, \quad (106)$$

where r represents each region in GTEM, while in MMRF, r represents Australia only; $a(r)$ is a positive parameter defining maximum number of permits which can be imported. Let $t(r)$ be the shadow price of this constraint. Then equations that need to be added to GTEM and MMRF to implement the above constraint are:

$$t(r) \cdot \{1 - [Emis(r) - Quota(r)] / [a(r) \cdot Quota(r)]\} = 0, \quad (107)$$

$$[Emis(r) - Quota(r)] / [a(r) \cdot Quota(r)] \leq 1, \text{ and} \quad (108)$$

$$t(r) \geq 0. \quad (109)$$

$t(r)$ is the additional price which emitters in region r need to pay if the constraint becomes binding. Because of this, the regional permit price needs to be modified as:

$$CTAX(r) = GLOBCTAX \cdot e(r) + t(r), \quad (110)$$

where $CTAX(r)$ is the regional permit price, $GLOBCTAX$ is the global permit price in international dollars, $e(r)$ is the exchange rate expressed as the units of local currencies per unit of international dollar, see Figure 17 for the relationship between the constrained trade of permits and the shadow price of the constraint.

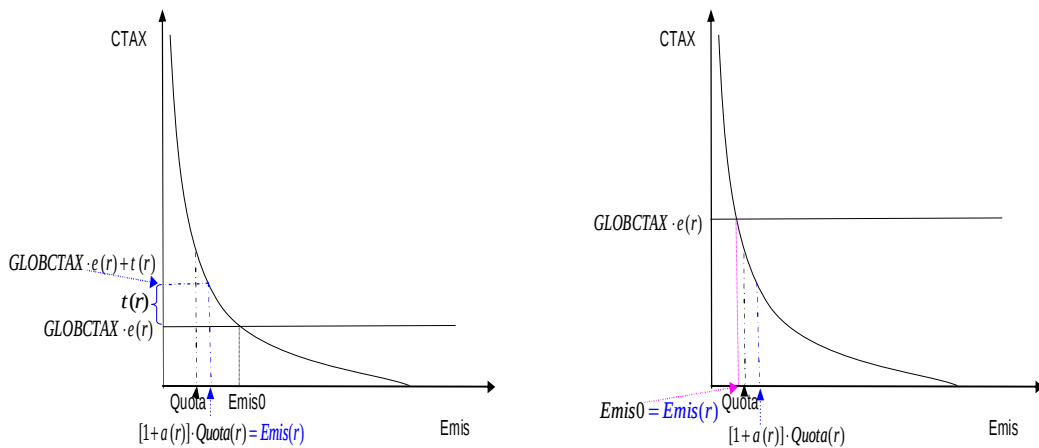


Figure 17. Illustrative diagram of quota constraint, emissions and shadow price of the constraint: left one showing the constraint is binding, while the right one the constraint is not binding.

The proposed method is then used to implement the equations (107) to (109) in GTEM and MMRF with perturbations and converting the inequalities to equalities in the same way as described in all earlier examples.

Shown in Table 4 are the results from GTEM showing the constraint's binding status and its associated shadow price for one of the GTEM regions: EU25 (European Union). It should be noted that a small $a(r)$ is assumed only for EU25 and Australia in the model for testing the inequality constraint, while for all other regions, $a(r)$ is chosen to be equal to 1, which is sufficiently large that did not lead to the constraint binding for any of these regions. The results on the movement between the shadow price and binding status of the constraint for Australia are exactly the same as those for EU25. As it can be seen from the results, the shadow price movement perfectly matches the constraint's binding status. This clearly shows that the method is easy to be implemented in an existing model and can successfully solve the model with inequality constraints.

Table 4. Results by GTEM with the emission constraint for the region of EU25

$r = EU25$	Year					
	1	2	3	4	5	6
$a(r)$	0.001	0.001	0.001	1	1	0.001
$t(r)$	2.02	2.03	2.08	0	0	1.96
$Qcons(r)$	1	1	1	0.01	0.01	1

where $Qcons(r) = [Emis(r) - Quota(r)] / [a(r) \cdot Quota(r)]$, and the shadow price, $t(r)$, is in 2001's local currency.

Shown in Table 5 are the results from MMRF showing the constraint's binding status and its associated shadow price. The shadow price movement perfectly matches the constraint's binding status. This again demonstrates the effectiveness of the proposed method in another existing model.

Table 5. Results by MMRF with the emission constraint for Australia

	Year						
	1	2	3	4	5	6	7
a	0.01	0.01	0.01	0.01	0.01	0.01	0.51
t	0	6.12	12.5	15.5	19.8	4.71	0
$Qcons$	0.7	1	1	1	1	1	0.09

where the shadow price is in 2005's Australian dollars.

It is worth to mention that implementation of the constraint of permit trading using the proposed method does not increase the computational time much in the above two models – this is in contrast to the implementation using the two pass method where not only the computational time is significantly increased but also it often fails to solve the models or gives the incorrect solution with the constraint.

The path of parameter a in the Tables 4-5 was arbitrarily chosen to see the constraint changing from (to) binding to (from) non-binding. The results clearly showed that the method can detect the change of binding status of the constraint. The high shadow price of the constraint in MMRF is due to the emission reduction scheme

implemented in the model. Detailed interpretation of the results is beyond the scope of this paper.

3.6.2. Constraints on rate of return on capital

In a recursive general equilibrium model with sector specific capital, the sector specific rate of return on capital could be negative. In this example, all sector specific rates of return are constrained to be non-negative using the proposed method. This constraint is implemented in the TCGE model by specifying the complementarity between the capital utilisation and the non-negative rate of return (see Figure 18 for the illustration of this complementarity condition).

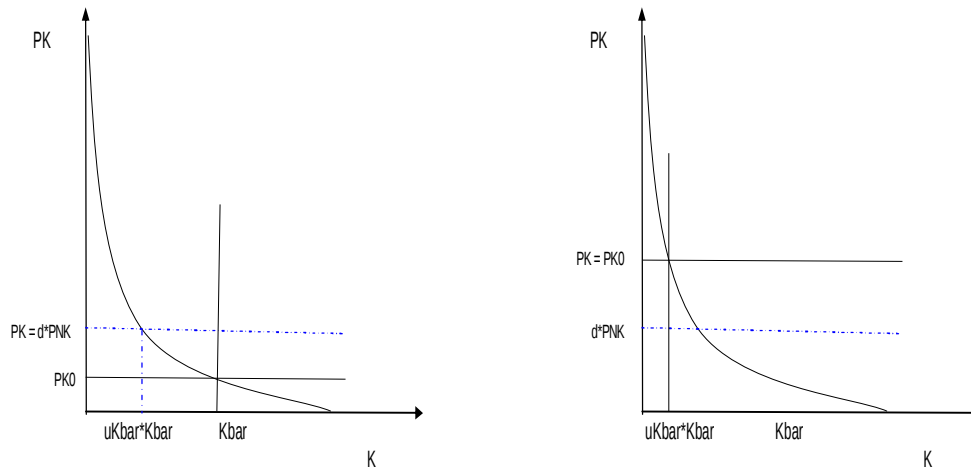


Figure 18. Illustrative diagram of constraint on the rate of return or the capital rental (PK) and the associated shadow price of the constraint: left one showing the constraint is binding, while the right one the constraint is not binding. All notations in the diagram are explained in the equations below.

The constraint of non-negative rates of return is specified mathematically in the TCGE as follows:

$$ROR(j, r) \geq 0, \text{ where } j \text{ represents each sector and } r \text{ represents each region,}$$

which is equivalent to by the definition of ROR in the TCGE:

$$PK(j, r) \geq d(j, r) \cdot PNK(j, r), \quad (111)$$

where $PK(j, r)$ is the capital rental, $d(j, r)$ is the depreciation rate of capital, and $PNK(j, r)$ is the cost of a unit of new capital.

Let $puKbar(j, r)$ be the shadow price of this constraint. Then equations that need to be added into TCGE to implement the above constraint are:

$$puKbar(j, r) \cdot \{PK(j, r) - d(j, r) \cdot PNK(j, r)\} = 0, \quad (112)$$

$$PK(j, r) \geq d(j, r) \cdot PNK(j, r), \text{ and} \quad (113)$$

$$puKbar(j, r) \geq 0. \quad (114)$$

$puKbar(j, r)$ enters into the model via linking to the utilisation rate of the capital (see Figure 18). That is,

$$uKbar(j, r) = 1 - puKbar(j, r), \quad (115)$$

$$K(j, r) = uKbar(j, r) \cdot Kbar(j, r), \quad (116)$$

where $Kbar(j, r)$ is the total capital stock available through the capital stock and investment accumulation, and $K(j, r)$ is the demand for capital.

The proposed method is then used to implement the equations (112) to (114) in the TCGE with perturbations and converting the inequalities to equalities in the same way as described in all earlier examples.

Table 6. Results by TCGE with a global emission price for the electricity technology of coal with the constraint on its rate of return on capital

Region		Year						
		1	2	3	4	5	6	7
AUS	<i>ROR</i>	0.05	0	0	0	0	0	0
	<i>uKbar</i>	1	0.82	0.81	0.79	0.77	0.75	0.73
USA	<i>ROR</i>	0.02	0	0	0	0	0	0
	<i>uKbar</i>	1	0.67	0.65	0.63	0.61	0.59	0.57
CHINA	<i>ROR</i>	0	0	0	0	0	0	0
	<i>uKbar</i>	0.92	0.89	0.88	0.86	0.84	0.83	0.81
ROW	<i>ROR</i>	0.13	0	0	0	0	0	0
	<i>uKbar</i>	1	0.91	0.89	0.86	0.84	0.81	0.78

where ROW represents the rest of world.

Shown in Table 6 are the results from TCGE showing the constraint's binding status and its associated shadow price for one of the sectors, i.e., the coal technology in the electricity sector. The results for all other sectors tell the exactly same story on the movement of the binding status and the shadow price. Again, as it can be seen from the results, the shadow price movement perfectly matches the constraint's binding status, where $uKbar = 1$ means the zero shadow price (i.e., $puKbar = 0$).

4. Conclusion and discussion

A new method is proposed to solve problems with inequality constraints or complementarity conditions using GEMPACK. The method is easy to implement. Adding an inequality constraint to an existing model, one just needs to express the constraint as a complementarity problem, and then perturb the problem and convert the inequalities to equalities as discussed in the method and in the examples. One also needs to make sure that the shadow price of the inequality constraint enters into relevant equations in the model. The remaining step is then to shock the perturbation to zero to get the true solution of the problem together with intended shocks on all other exogenous variables.

The effectiveness of the method has been clearly demonstrated by all examples presented in the paper. A minor implementation issue is the 'arbitrary' choice of the

initial perturbation (i.e., the initial value of e in the examples). In fact, in most of the examples that have been tested including those not presented in this paper, the perturbation can be initialised to a reasonably small but arbitrary number – generally, the method works quite well to have the perturbation initialised to any number between 0.001 and 0.1. However, there are examples such as the perfect substitution example presented in this paper (for the general equilibrium example in this paper, it is also assumed perfect substitution of the outputs by the two technologies) where there are sudden switches of solutions between far different alternatives (e.g., production by a technology goes to zero immediately from a big number when its production cost exceeds its competitors); these types of problems are really hard for GEMPACK to solve due to its linearization-based solving approach – for these cases, a few trial runs may be needed to get a proper initial perturbation where a relatively large value of the perturbation, say, between 0.1 and 2, may be used for the trials. Even for these hard examples, the proposed method works extremely well, as it is demonstrated by the perfect substitution example as well the general equilibrium example presented.

There is also a trade-off issue. A good perturbation requires less number of sub-intervals to get reasonably accurate solution. On the other hand, even with a “bad” perturbation, one may still get reasonably accurate solution by using the large number of sub-intervals. Of course, it also depends on the sizes of other shocks. A small number of sub-intervals may be used to get less accurate solution to save computational time – where the solution may not hit the bound exactly but with reasonably small errors.

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